DISTRIBUTED ADAPTIVE SIGNAL PROCESSING: A STUDY OF SERIES FEED-FORWARD BINARY ADAPTIVE COMPOUNDS

A Thesis

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ABSTRACT

This work attempts to characterize and provide design guidelines for adaptive systems composed of two adaptive elements. To begin, we introduce the notion of an adaptive element, which is the smallest scale upon which systems have adaptivity. Then we provide a concise review of many of the subjects deemed important to single adaptive elements from a deterministic dynamical systems view. Theorems are provided characterizing the deterministic dynamic behavior of single adaptive algorithms, and their robustness to disturbances and time variation. We also provide a review of deterministic single time scale averaging theory.

After we have characterized the behavior of a single adaptive element, we are ready to begin to study the possibility of connecting more than one adaptive device together. It is this possibility that inspires us to imagine different ways of and reasons for using distributed adaptation in these structures. We then select one of the possible binary (i.e., two element) structures for connecting adaptive elements, the series feedfoward binary adaptive compound (SFFBAC), and characterize its behavior. We motivate this discussion with quotes from the digital receiver literature, which indicate that the interaction of adaptive components is a recognized problem about which very little theoretical work has been done. All along, we are considering adaptive systems from an engineering mindset. Within these lines, we observe that distributed adaptive systems often arise as a relic of our method of design. We name the design technique that connects individually designed adaptive elements together to solve a bigger problem the "Divide and Conquer" strategy. Our goal then becomes to provide sufficient conditions under

which we can use "Divide and Conquer" to design working series feed-forward binary adaptive compounds.

To give conditions for Divide and Conquer design, we begin with a qualitative mindset, describing the sorts of requirements that may be encountered when applying a more rigorous theorem. Then, in Chapter 4 we develop the beginning of a rigorous behavior theory for series feed-forward binary adaptive compounds. This theory directly supports the qualitative design conditions we provide. Since we are not only interested in studying adaptive receivers which behave well, we also develop a misbehavior theorem, which predicts one of the ways a SFFBAC may misbehave. We conclude this theory by applying it to practical examples from digital receivers employing more than one adaptive element. To do so, we must remove the assumption that all other adaptive elements are behaving in an ideal manner from the models for the behavior of the adaptive receiver components. Thus, Appendix A contains models and derivations for the behavior of adaptive receiver components under non-ideal situations. After our example applications of the theory to adaptive receivers, we end the thesis by mentioning other possible applications of the theory, as well as many possible extensions to the theory, including the use of other mathematical tools.

BIOGRAPHICAL SKETCH

John Walsh was born in Carbondale, Illinois on September 23, 1981. He attended elementary school in Carbondale, Ill, and later in Virginia Beach, VA. He attended middle and high school at Norfolk Academy (Norfolk, VA.) where he learned a love for theater and German. He received the Bachelor of Science Degree magna cum laude from Cornell University in 2002. Since then he has continued his studies at Cornell University in pursuit of a MsPhD degree. He is a member of Tau Beta Pi, and Eta Kappa Nu, and he loves to play the guitar.

To my family.

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I especially want to thank my Mother, Father, and sister for their unfailing support and encouragement throughout my life. Thanks and love also go to Grandpa John who showed me the joy and benefits of true hard good work, and to Gammy, Grandma Bets, Grandpa Bill, Uncle Arthur and Aunt Edith. I also would like to thank my two officemates, Andy Klein and Rick Martin, who have shared many an enjoyable day shouting "Matlab" down the hall. Cyprian Wejnert and Domonic Leparie have been constant friends during my stay in Ithaca, and I appreciate their friendship greatly.

In terms of intellectual influences, I definitely have to thank Bill "You're a professor" Sethares for his patient help with many a technical discussion, as well as his excellent ability as a bassist. I wish to thank Lang Tong for teaching me the fundamentals of digital communications, detection, and estimation in such an insightful and exciting manner. Although their influence is not directly noticeable here, these classes laid an excellent theoretical foundation for my future research. I want to thank the Cornell University ECE Department for funding me as a teaching assistant during a good portion of the time I was working on the research for this thesis, and the National Science Foundation which funded this research in part.

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Chapter 1

Introduction

Adaptive systems are ubiquitous in our world. They occur in economics, in populations; in networks and in social organizations. Most biological systems adapt to their environment, and hence contain adaptive aspects. We are ourselves adaptive systems: the brain being one of the most complex adaptive systems imaginable. It is not surprising, then, that we design adaptivity into what we build.

Communications, control, and signal processing are all fields of engineering that rely heavily on adaptive systems for practical implementations. Usually, these systems can be viewed as complex interactions between smaller adaptive objects. To borrow an example from economics, a national economy is certainly a large adaptive system, in which smaller players, in the form of businesses and governments, interact in complex ways. When investigating the economy as a whole, these individual player's actions are almost imperceivable in terms of the way they can affect the large system. Yet, when viewed on their own scale these smaller players, the businesses and governments, are adaptive systems themselves.

Continuing a 2,500 year old Platonic philosophical trend, we hypothesize that there is a smallest scale upon which adaptation occurs. In this manner, the economy can be likened to a block of matter. Just as when looking at a block of matter, it is difficult to discern the existence or effect of a single atom, it is difficult to discern the effect, or even the existence, of a single business when studying the behavior of the entire the economy. Yet, Greek philosophy chose to attack a characterization of the behavior of matter by first hypothesizing the existence of, and then characterizing, the smallest scale upon which matter could be described.

Similarly, we wish to attack the problem of understanding complex adaptive systems by first understanding their behavior on the smallest scale possible. Thus, we introduce the notion of an adaptive element: the smallest building block (ie type of atom) in a compound adaptive system. Adaptive elements are the fundamental particles of an adaptive system: below the scale of an adaptive element, the subsystems are no longer adaptive.

In this thesis our scope is limited further in that we are only going to consider adaptive elements that can be viewed from a signal processing standpoint. These adaptive signal processing elements take an input signal and produce an output in a manner which changes depending on characteristics of the inputs and previous outputs. Furthermore, we will limit our discussion to adaptive signal processing elements that operate in the discrete domain. That is, they take in samples at discrete time instants, and process them to create more samples at discrete time instants.

Digital adaptive signal processing elements have existed for decades now, and a wide base of knowledge has been collected about both individual algorithms, as well as general algorithm classes. Rather than collect or imagine that we could treat the entirety of this literature with any type of adequacy here, we choose to introduce some fundamental mathematical principles upon which an adaptive element can be built, and many adaptive elements are built. We will draw many of our examples of adaptive elements from communications systems: a technical area in which adaptive widgets have proven their applicability and efficiency in a wide range of products and applications. But 40 years of research on single adaptive algorithms, numerous Bible-sized textbooks, and a gigantic body of journal literature have all probably begun to give single adaptive algorithms somewhat of an adequate treatment, so we do not presume we will say anything new about that subject here. Everything this thesis presents about single adaptive elements should be easy to find in any number of more extensive textbooks on the subject (e.g. [1], [2], or [3]). The material is included here to motivate the heart of our discussion, which centers around the possibility of interconnecting two adaptive elements. This is our first step towards characterizing the behavior of larger adaptive systems.

Clearly, there are phenomena that will occur in complex adaptive systems which will not be present in a small adaptive compound made of two elements. Nevertheless, we hope to begin to learn about the larger system by attacking a small system containing only a few adaptive elements first, just as it is common practice to learn inorganic chemistry with small molecules before attempting to study gigantic organic compounds. Furthermore, as we will see in Chapter 4, even after constraining our study to the behavior of adaptive compounds formed from only two adaptive elements, a.k.a. Binary Adaptive Compounds (BACs), the possibilities of interconnecting the adaptive elements are numerous. Thus this thesis attempts to discuss only one, and perhaps the simplest, type of Binary Adaptive Signal Processing Compound: a compound in which the two adaptive elements are connected in series, each adapting only on their own output and input, and the output of the first being fed into the input of the second.

Even within this very narrow context our coverage is probably not comprehensive, since in this early (masters) work, we seek theorems with only sufficient, but perhaps not necessary, conditions. Nevertheless, we attempt to provide some significant insights into the way such series feed-forward binary adaptive signal processing compounds behave. Specifically, we provide:

- A theorem which gives sufficient conditions on the two adaptive elements in the compound to guarantee convergence to fixed desired parameter settings.

- Theorems which account for non-idealities in the two adaptive elements, including time variation, disturbances, and un-averaged behavior, yet still guarantee convergence to parameter settings near the desired trajectories.

- A collection of sufficient conditions, the Divide and Conquer Conditions, which will guarantee proper operation of a series feed-forward binary adaptive compound when it is designed within a particular framework.

- A misbehavior theorem, which emphasizes the importance of choosing proper step sizes to track time-varying desired trajectories, and shows the lack of convergence that is possible when they are not chosen correctly.

We will see that these results have some applications in communications systems, where it is common practice to interconnect several adaptive elements in a heuristic manner. Our hope is that our theorems and discussion will be able to illuminate this one type of interconnection in an adaptive communications receiver and provide insightful implications for design and adjustment. Along these lines, we end the thesis using our theorems to analyze several examples of algorithm pairs found in digital receivers. To do so, we must determine the true behavior of different receiver components when their inputs are non-ideal. We call the functions which determine this behavior ”sensitivity functions” and have developed them for a variety of adaptive receiver components in Apendix A. With these new models, we use our averaging theory to analyze an interconnected timing recovery algorithm and equalization algorithm. We also apply our theory to two adaptive receivers containing a gain control and different carrier recovery units to show how the theorems we have developed can help us decide between two adaptive algorithms. We conclude the thesis by mentioning some other possible applications of the theory we have developed, and by noting a number of possible directions in which to extend the theory, including the possible use of other mathematical tools.

Chapter 2

What is an adaptive element? An Adaptive Element Characterization.

Adaptive elements are the fundamental building blocks of adaptive systems. We consider them to be the smallest scale upon which a part of an adaptive system can be viewed as performing adaptation. In context of signal processing, adaptive signal processing elements[1] take an input signal and produce an output in a manner which changes depending on characteristics of the inputs and previous outputs. Of course, this general definition does not provide us with a unique description of an adaptive element, since an interconnection of two adaptive elements satisfying this definition will create another system that satisfies this definition. We do not address this problem of non-uniqueness here, other than to mention it, and to note that we will divide the composite adaptive system up into adaptive elements along conventional lines (i.e., an equalizer and a timing recovery unit are two separate adaptive elements in a digital receiver).

2.1 Composition: The Subatomic Particles

From the discussion in the previous chapter, we can discern two tasks which an adaptive element must perform:

1. Process the input to create the output and adapt, and

2. Adapt its method of processing the input to create the output based on

[1]From this point on we will refer to adaptive signal processing elements using the phrase "adaptive elements."

previous inputs and outputs.

Mathematically speaking, this separation into tasks suggests the idea that there are two subsystems of equations involved with an adaptive element. Continuing the analogy with chemistry, we call these two subsystems the subelements or subatomic particles. The *processing subelement* processes the input signal to create an output signal, and the *adaptation subelement* determines how it should do so. Because we wish to be able to describe the adaptive element in a mathematical manner, we assume that the communication between the adaptive subelement and the processing subelement occurs in the form of a parameter vector, which we call the ***adaptive state***. The adaptive state contains all memory the adaptive element has of past inputs[2] and totally determines the manner in which the adaptive element processes the current input to create the current output. The right hand pane of Figure 2.1 emphasizes the separation of the adaptive structure into two substructures, one that controls the adaptation by changing the adaptive states, and one that creates the input from the output based on the adaptive state. The left pane shows a diagram that is equivalent, yet more compact, which we will use from this point on.

2.1.1 Adaptatation Sub-element

The adaptation sub-element is what differentiates an adaptive signal processing element from a simple filter. It controls the evolution of the adaptive state based upon the inputs its observes. Because the adaptive state contains all the memory

[2]Thus, for an adaptive IIR filter, what we are calling the "adaptive state" is the concatenation of both the parameters as well as the current state from the state space description of the filter. Any separating of these two vectors (e.g. for mixed time scale analysis) will be explicitly dealt with when necessary.

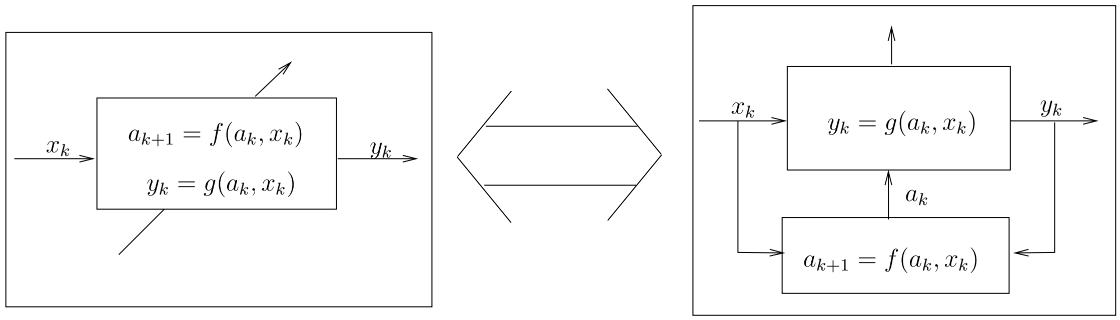

Figure 2.1: Two equivalent representations of an adaptive element.

the adaptive device has of the past, the new adaptive state is totally determined by the current adaptive state and the current input. Thus, the evolution of the adaptive state vector can be described with the difference equation:

$$a_{k+1} = f(a_k, x_k)$$

Where, as in Figure 2.1, $a_k \in \mathbb{R}^n$ is the adaptive state at positive integer time instant k and $x_k \in \mathbb{R}^P$ is the input vector at time k. In the right pane of Figure 2.1, the adaptation subelement lies within the lower of the two boxes.

2.1.2 Processing Sub-element

Given a particular adaptive state, the processing sub-element processes the input to create the output. This mapping it totally determined once a particular adaptive state is specified. Thus, this subsystem of equations can be written as

$$y_k = g(a_k, x_k)$$

where $x_k \in \mathbb{R}^p$ is the input, $a_k \in \mathbb{R}^n$ is the adaptive state, and $y_k \in \mathbb{R}^Q$ is the output of the adaptive signal processing element at time k. In the right pane of Figure 2.1, the processing element lies within the higher of the two boxes.

2.2 Designing Adaptive Elements and Characterizing their Behavior

Now that we have determined what an adaptive element is, we move on to discuss different ways of designing adaptive elements and characterizing their behavior. Before we can do so, we must make an important stipulation on the modelling of the signals involved with the adaptive element. Specifically, we are interested in signals which are specified within a deterministic[3] framework. This being said, we still allow uncertainty into our models by adding a disturbing signal which we do not directly specify other than to say it is bounded. Furthermore, our discussion of averaging can be coupled with notions of probabilistic expectation, so as to include many of the more important concepts that a stochastic analysis can bring. This allows a certain amount of marriage between the deterministic and stochastic theories, while still keeping some of the nicer aspects of a truly deterministic theory, such as the ability to bound errors concretely before limits are taken.

Once we have specified that we are interested in studying adaptive elements within a deterministic framework, there are a number of different theoretical fields which offer us insights into the way adaptive devices behave. Specifically, stability theory allows us to characterize the behavior of (possibly nonlinear) adaptive devices near an equilibrium. This is coupled with a common philosophy encountered when one designs adaptive elements. Typically, designers wish to perform some action on a signal which depends on the particular signal being encountered (hence the choice of an adaptive device). They come up with a function that gives a "good" output, given a particular input. Then, they choose a parameterized in-

[3]That is, non-random and non-stochastic.

put output relationship, which, when given the correct parameters for a particular input, produces the desired output. This parameterized relationship is then chosen to be the processing sub-element, and the job of the adaptation sub-element is to infer the "good" parameters from the current input. A local characterization of the adaptive element's behavior can then be performed by studying the dynamics of the adaptive subsystem near the "good" parameters for the class of inputs of interest. Specifically, it is desirable that, given a particular input, an adaptive subsystem initialized with an adaptive state near the "good" parameters will remain close to the "good" setting. The language of stability theory allows us to more precisely describe such behavior. Below are some definitions from the stability theory of difference equations that we will use in the ensuing discussion. Note that standard stability theory includes many more types of stability, but we list only those that we will study in the ensuing discussion. Definitions 1 through 4 can be found in almost any book on difference equations (e.g. [4] or [5]) or nonlinear systems (e.g. [6], [7], and [8] for continuous time versions). Although we use the notation k_0 for the notion of an initial time here, from now on, we will consider only systems which have the convention of starting at time 1, that is, with $k_0 = 1$. Note that definition 5 is a particular type of uniform ultimate boundedness, and is our own combined notion of stability and boundedness that we will have occasion to use in this thesis.

Definition 1 (Stationary Point). *A stationary point of a map* $f(k, \cdot, x_k) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ *is a point* $a^* \in \mathbb{R}^n$*, possibly dependent on* k *and* x_k*, such that* $a^* = f(k, a^*, x_k)$*.*

Definition 2 (Exponential Stability). *Given an input,* x_k*, a stationary point* a^* *of a map* f *in a difference equation* $a_{k+1} = f(k, a_k, x_k)$ *is said to be exponentially stable if there exists* $\delta > 0$*,* $M > 0$*, and* $\eta \in (0, 1)$ *such that* $\|a_k(k_0, a_1) - a^*\| \leq$

$M\|a_1 - a^*\|\eta^n$, whenever $\|a_1 - a^*\| < \delta$.

Definition 3 (Global Exponential Stability). *Given an input, x_k, a stationary point a^* of a map f in a difference equation $a_{k+1} = f(k, a_k, x_k)$ is said to be globally exponentially stable if it is exponentially stable with $\delta = \infty$.*

Definition 4 (Uniform Ultimate Boundedness). *A solution, $a_k(k_0, a_1)$ to the difference equation $a_{k+1} = f(k, a_k, x_k)$ is said to be uniformly ultimately bounded if there exists $\delta > 0$ and $M > 0$ such that $\lim_{k\to\infty} \|a_k(k_0, a_1) - a^*\| < M$ whenever $\|a_1 - a^*\| < \delta$ and M is independent of k_0.*

Definition 5 (Exponential Stability to within a Ball). *A solution, $a_k(k_0, a_1)$, is exponentially stable to within a ball of size M with rate α, if there exists δ, M and $\alpha \in (0,1)$ such that $\|a_k(k_0, a_1) - a^*\| \leq \alpha^{k-1}\|a_1 - a^*\| + M$ whenever $\|a_1 - a^*\| \leq \delta$.*

Note that, from now on, we will use the short hand a_k to refer to a particular solution to the system of equations, and drop the notation $a_k(k_0, a_1)$, which emphasized the explicit dependence of the trajectory on its initial location a_1, and the starting time k_0. We will also use the convention throughout that the initial starting time $k_0 = 1$.

2.2.1 Stability of Fixed Stationary Points

Now that we have familiarized ourselves with some of the important terms, let us begin characterizing some typical behavior of adaptive elements. In particular, we expect that when an adaptive element is excited with an input in the class of inputs of interest that is free from any disturbances, that it will eventually go to the correct desired state, preferably exponentially fast. Given a particular input our first theorem provides sufficient conditions upon the adaptive state equation and state initialization to guarantee exponential stability to the desired parameter settings.

Theorem 1 (Exponentially Stable Adaptive Elements). *Consider the adaptive state equation of an adaptive element*

$$a_{k+1} = f(a_k, x_k)$$

Suppose the transition function has a stationary point a^*, *such that*

$$a^* = f(a^*, x_k) \qquad \forall k \tag{2.1}$$

Furthermore, suppose that f *is locally contractive uniformly in* x_k *within an open ball* $\mathcal{B}_{a^*}$ *at* a^*

$$\mathcal{B}_{a^*} = \{a | \|a - a^*\| < r\}$$

$$\|f(\xi, x_k) - a^*\| < \alpha \|\xi - a^*\| \qquad \forall \xi \in \mathcal{B}_{a^*} \quad \forall k \tag{2.2}$$

Then, if the contraction constant α *is less than 1, the adaptive element is locally exponentially stable to* a^* *with rate* α *within a ball* $\mathcal{B}_{a^*}$

$$\alpha < 1 \Rightarrow \textit{L.E.S. to } a^*$$

$\diamond$ Start with the state equation

$$a_{k+1} - a^* = f(a_k, x_k) - a^*$$

Using (2.1) we have

$$a_{k+1} - a^* = f(a_k, x_k) - f(a^*, x_k)$$

Taking the norm of both sides and using (2.2) we have

$$\|a_{k+1} - a^*\| \leq \alpha \|a_k - a^*\|$$

Running the recursive formula gives

$$\|a_{k+1} - a^*\| \leq \alpha^k \|a_1 - a^*\| \quad \forall a_1 \in \mathcal{B}_{a^*}$$

which is a definition of exponential stability. $\diamond$

Commentary and Examples

A subtle point that the mathematics do not highlight is the possible dependence of α on the input. Thus, one of the things one must check when applying such a theorem to an adaptive element is that $\alpha < 1$ for all of the possible inputs of interest. To explain the meaning of the theorem, here are a couple of mathematical examples of adaptive state equations after specifying a particular input that are exponentially stable to the desired parameter settings.

Example 2.1 (An Exponentially Stable Adaptive Element). *Here we have an exponentially stable adaptive element with adaptive state equation*

$$a_{k+1} = a_k - \mu^2 a_k + \mu a_k^3$$

from which we see that $a^* = 0$ *is a stationary point. Let's consider the ball,* $\mathcal{B}_{a^*} = \{a | \|a - a^*\| < 1\}$. *Within this ball our contraction constant will be*

$$\begin{aligned} \alpha &= \max_{a \in [-1,1]} \tfrac{|a - \mu^2 a - \mu a^3|}{|a|} \\ &= \max_{a \in [-1,1]} \tfrac{|a||1 - \mu^2 - \mu a^2|}{|a|} \\ &= \max_{a \in [-1,1]} |1 - \mu^2 - \mu a^2| \\ &= 1 - \mu^2 \end{aligned}$$

which will be $< 1 \quad \forall |\mu| < 1$. *Thus, the theorem predicts that our algorithm will be exponentially stable within* $\mathcal{B}_{a^*}$ *for all* $|\mu| < 1$. *Figure 2.2 suggests that this is indeed the case. In the top left corner of this figure, we see the adaptive state*

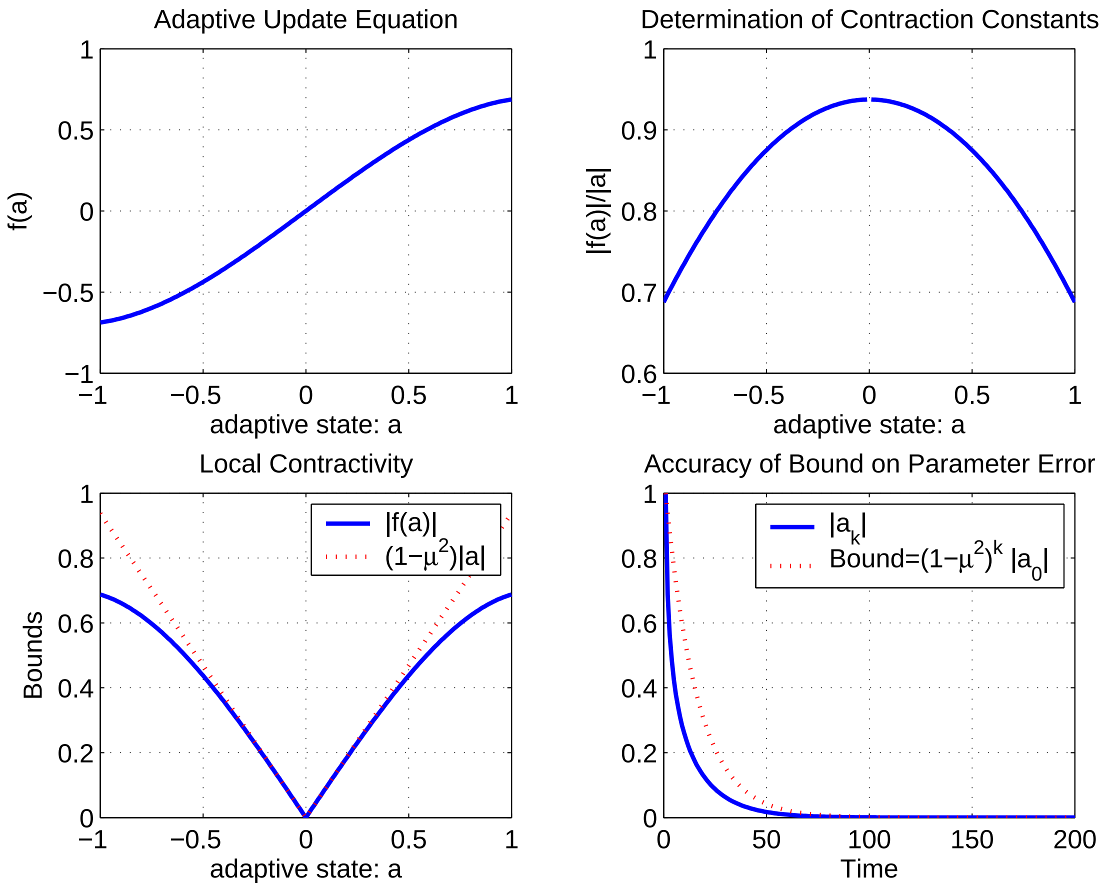

Figure 2.2: A sample application of theorem 1, indicating exponential stability of locally contractive adaptive state update equations.

function, and in the top right corner we see the graph that helps determine the contraction constant for the particular region, $[-1, 1]$, *that we are interested in. The bottom left pane shows the accuracy of our contraction bound, and we observe that as the error in the state gets larger our bound becomes less tight. One can discern from the bottom right hand graph that the algorithm converges faster than our worst case bound predicts.*

It is tempting to replace the contraction criterion, (2.2), with a stipulation that the adaptive state update equation is Lipschitz continuous[4] with Lipschitz constant less that one. Specifically, this requirement takes the mathematical form

$$\|f(a_1) - f(a_2)\| \leq \alpha \|a_1 - a_2\| \quad \alpha < 1 \quad \forall a_1, a_2 \in \mathcal{B}_{a^*} \tag{2.3}$$

While this does guarantee exponential stability to the desired point, it is too restrictive. The two examples below show that the less restrictive (2.2) is sufficient to guarantee exponential stability: one does not need to require (2.3).

Example 2.2 (Discontinuous Exponential Stability). *In this example, we consider an adaptive state equation which is discontinuous in the adaptive state, but still satisfies the contraction criterion that guarantees exponential stability in Theorem 1. Specifically, the adaptive state function is*

$$f(a) = \begin{cases} \frac{a}{10} & |a| < .2 \\ \frac{9a}{10} & |a| \geq .2 \end{cases} \tag{2.4}$$

While another possible adaptive state function that emphasizes the lack of necessity of (2.3) is

$$f(a) = \begin{cases} \frac{a}{10} & |a| < .2 \\ 100\left(x_k - .2 sign(x_k)\right) + .02 sign(x_k) & .2 < |a| < .21 \\ \left(x_k - .201 sign(x_k)\right)/10 + .12 sign(x_k) & |a| \geq .21 \end{cases} \tag{2.5}$$

Both of these adaptive subelement's behaviors are plotted in Figure 2.3. The update function in the top left pane is (2.4) and is discontinuous, and the lower left update function is (2.5) and has a Lipschitz constant that is greater than 1. Yet, both systems are asymptotically stable, as seen by the example trajectories shown in the right column of Figure 2.3. This gives us an indication of the generality of our assumptions compared to other possible assumptions, while it turns out that they are still only sufficient conditions.

[4]See Appendix C for a definition of Lipschitz continuity.

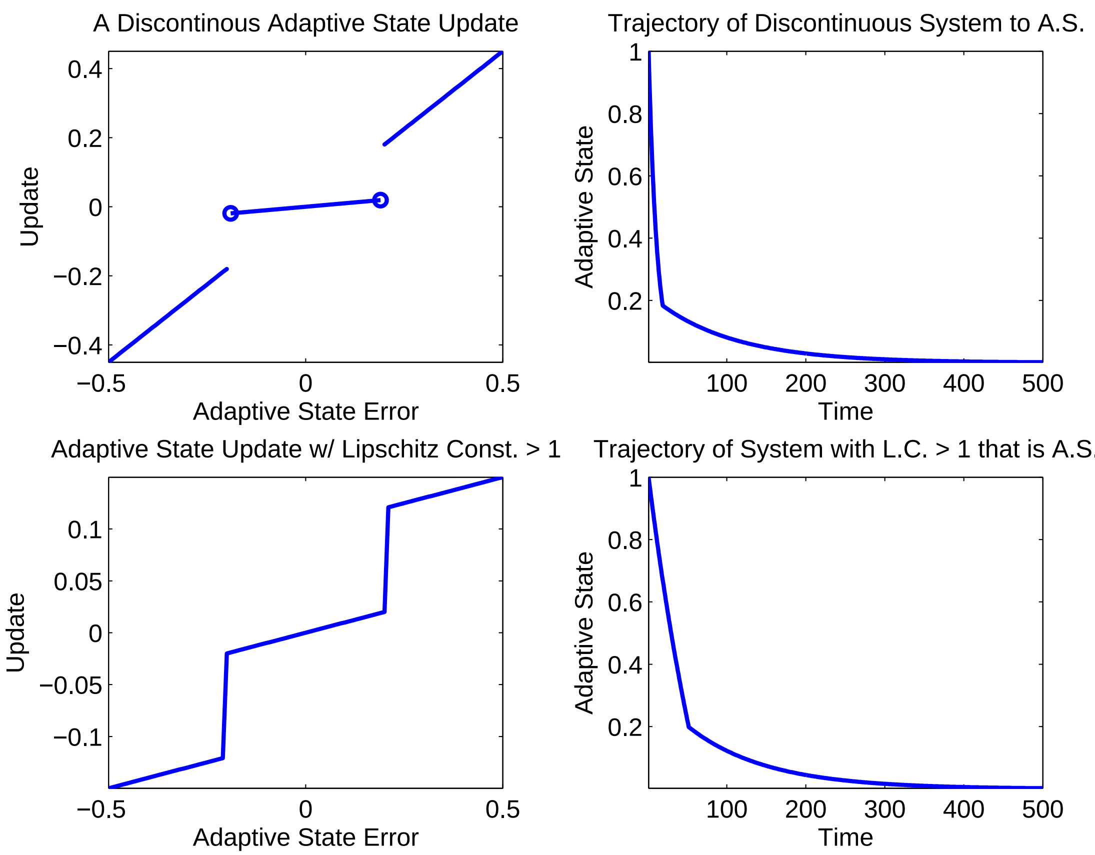

Figure 2.3: Examples of asymptotic stability with a discontinuity or a Lipschitz constant greater than one.

It is also fairly important to note that our theorem only provide sufficient conditions for exponential stability to the desired parameters. Even though our conditions are more general than they could be, they are still not general enough to be necessary.

2.2.2 Robustness of Exponential Stability to Perturbations

One very important property of exponential stability is the robustness to disturbances it induces. Namely, when either our input or our adaptive state equation has modelling error in it which is uniformly bounded, the adaptive element is still stable to a ball around its desired performance settings whose size is linearly related to the bound on the disturbance. Theorem 2 emphasizes and proves this property of exponential stability under the conditions we provided in Theorem 1.

Theorem 2 (Robustness of Exponential Stability to Disturbances). *Consider a disturbed (noisy) adaptive state equation of an adaptive element*

$$a_{k+1} = f(a_k, x_k) + n_k$$

Such that the undisturbed transition function is contractive and thus locally exponentially stable to a point a^* within an open ball $\mathcal{B}_{a^*}$ with a rate α

$$\mathcal{B}_{a^*} = \{a | \|a - a^*\| < r\}$$

$$a^* = f(a^*, x_k) \qquad \forall k \tag{2.6}$$

$$\xi_{k+1} = f(\xi_k, x_k) \quad \textit{and} \quad \xi_1 \in \mathcal{B}_{a^*} \Rightarrow \|\xi_{k+1} - a^*\| \leq \alpha^k \|\xi_1 - a^*\| \tag{2.7}$$

Then, if the disturbance and initial error are uniformly bounded, such that

$$\|n_k\| < c, \qquad \|a_1 - a^*\| + \frac{c}{1-\alpha} < r \tag{2.8}$$

the perturbed system remains within $\mathcal{B}_{a^*}$, and the adaptive element will be exponentially stable to a smaller ball surrounding a^* with radius $\frac{c}{1-\alpha}$.

$$\|a_{k+1} - a^*\| \leq \alpha^k \|a_1 - a^*\| + \frac{c}{1-\alpha}$$

◇ We start with the state update equation

$$a_{k+1} = f(a_k, x_k) + n_k$$

Subtracting a^* from both sides yields

$$a_{k+1} - a^* = f(a_k, x_k) - a^* + n_k$$

Taking norms of both sides, and using the triangle inequality yields

$$\|a_{k+1} - a^*\| \leq \|f(a_k, x_k) - a^*\| + \|n_k\|$$

Using the fact that a^* is a fixed point (2.6), and using the contractivity, (2.7) gives

$$\|a_{k+1} - a^*\| \leq \alpha \|a_k - a^*\| + \|n_k\|$$

Running the recursion yields

$$\|a_{k+1} - a^*\| \leq \alpha^k \|a_1 - a^*\| + \sum_{i=0}^{k-1} \alpha^i \|n_{k-i}\|$$

Using the boundedness of the disturbance (2.8), we have

$$\|a_{k+1} - a^*\| \leq \alpha^k \|a_1 - a^*\| + c \sum_{i=0}^{k-1} \alpha^i$$

Then, using the sum of an infinite geometric series gives

$$\|a_{k+1} - a^*\| \leq \alpha^k \|a_1 - a^*\| + \frac{c}{1-\alpha}$$

The last concern as to accuracy is that the perturbation never took us out of the ball, $\mathcal{B}_{a^*}$, in which the undisturbed system was exponentially stable, (2.7).

$$\|a_1 - a^*\| + \frac{c}{1-\alpha} < r$$

which (though overly conservatively) yields the last condition, (2.8), in the theorem. $\diamond$

Example 2.3 (Including Disturbances). *In this example, we consider the same system within the same ball as in Example 2.1. However, the adaptive state equation has a disturbance in it such that it is now*

$$a_{k+1} = a_k - \mu^2 a_k + \mu a_k^3 + .01\cos(\frac{2\pi k}{40})$$

We see that the maximum disturbance at any step is $c = .01$*, which indicates that the trajectory should converge to a ball of size* $\frac{.01}{1-\alpha} = \frac{.01}{\mu^2}$*. Figure 2.4 suggests that this is indeed the case. The bottom graph compares the actual norm of the trajectory with our bound. We see that the bound is very conservative, because we haven't dealt with the fact that the disturbance may be zero on average, as it is in this case as shown in the top figure. The bound assumes a worst case disturbance and a minimum contraction (given the assumptions) at every step, which explains why it is much larger in the bottom graph that the actual trajectory.*

2.2.3 Time Varying and Perturbed Adaptive Systems

We now extend our treatment to adaptive elements for which the optimal processing settings change over time. This situation typically occurs when the input signal has some sort of non-stationary property which affects what our optimal processing is. We see that, as long as the optimal adaptive state moves slowly enough, we can track the time variation to a specified accuracy. Furthermore, we can control the size of the ball around the optimal locations in which we lie by changing the decay rate of the original exponentially stable system. Theorem 3 concretely bounds the tracking error.

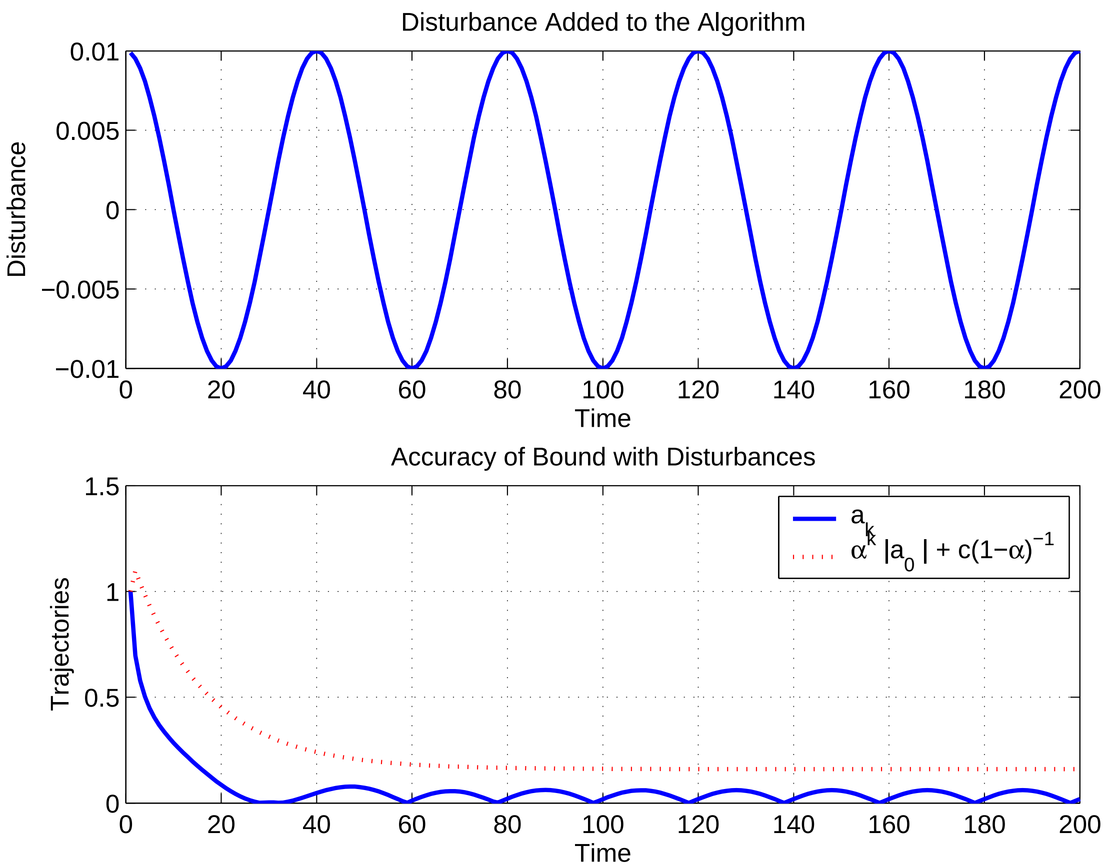

Figure 2.4: An example showing the accuracy of the bound provided by theorem 2.

Theorem 3 (Time Variation and Disturbances). *Consider a disturbed (noisy) time-varying adaptive state equation of an adaptive element*

$$a_{k+1} = f(k, a_k, x_k) + n_k \tag{2.9}$$

such that the undisturbed system has a time-varying stationary point at a_k^*

$$a_k^* = f(k, a_k^*, x_k) \quad \forall k \tag{2.10}$$

and that the undisturbed adaptive state update equation is contractive towards a_k^* *in a tube,* $\mathcal{B}_{a_k^*}$*, surrounding the equilibrium trajectory* a_k^* *uniformly in time.*

$$\mathcal{B}_{a_k^*} = \{\xi_k | \|\xi_k - a_k^*\| < r\}$$

$$\|f(k, \xi_k, x_k) - a_k^*\| < \alpha \|\xi_k - a_k^*\| \quad \alpha < 1 \quad \forall \xi_k \in \mathcal{B}_{a_k^*} \quad \forall k \tag{2.11}$$

Furthermore, suppose that the time variation is bounded uniformly

$$\|a_{k+1}^* - a_k^*\| < \gamma \quad \forall k$$

and that the disturbance is uniformly bounded

$$\|n_k\| < c \quad \forall k$$

Then, if the time variation is slow enough and the disturbance is small enough such that

$$\|a_1 - a_1^*\| + \frac{\gamma + c}{1 - \alpha} < r$$

Then, the disturbed time varying system is exponentially stable to a tube of radius $(1 - a)^{-1}(\gamma + c)$ *surrounding the equilibrium trajectory with rate* α

$$\|a_{k+1} - a_{k+1}^*\| < \alpha^k \|a_1 - a_1^*\| + \frac{\gamma + c}{1 - \alpha}$$

$\diamond$ We begin by adding a clever form of zero to (2.9)

$$a_{k+1} - a^*_{k+1} = f(k, a_k, x_k) - a^*_k + n_k + a^*_k - a^*_{k+1}$$

Using the triangle inequality, and the fact that a^*_k is a stationary point, (2.10), we have

$$\|a_{k+1} - a^*_{k+1}\| \leq \|f(k, a_k, x_k) - f(k, a^*_k, x_k)\| + \|n_k\| + \|a^*_k - a^*_{k+1}\|$$

Now, using the Lipschitz continuity, (2.11), we have

$$\|a_{k+1} - a^*_{k+1}\| \leq \alpha\|a_k - a^*_k\| + \|n_k\| + \|a^*_k - a^*_{k+1}\|$$

Running the recursion gives[5]

$$\|a_{k+1} - a^*_{k+1}\| \leq \alpha^k\|a_1 - a^*_1\| + \sum_{i=0}^{k-1} \alpha^{k-i} \left(\|n_k\| + \|a^*_k - a^*_{k+1}\|\right) \tag{2.12}$$

Using the boundedness of the time variation and disturbances and the sum of an infinite geometric series yields

$$\|a_{k+1} - a^*_{k+1}\| \leq \alpha^k\|a_1 - a^*_1\| + \frac{\gamma + c}{1 - \alpha}$$

which gives us the bound that the theorem states. We should also be certain that the trajectory never left the ball that the contraction condition was valid in

$$\alpha^k\|a_1 - a^*_1\| + \sum_{i=0}^{k-1} \alpha^{k-i} (c + \gamma) < r \quad \forall k$$

Using the sum of a finite geometric series yields

$$\alpha^k\|a_1 - a^*_1\| + \frac{1 - \alpha^k}{1 - \alpha} (c + \gamma) < r \quad \forall k$$

which we can absolutely guarantee (and then some) if

$$\|a_1 - a^*_1\| + \frac{c + \gamma}{1 - \alpha} < r$$

which is the bound, (3), required by the theorem. $\diamond$

[5]This is a good bound if you know the exact form of the disturbance and the exact a^*_k.

Example 2.4 (Time Variation and Disturbances). *Consider the following adaptive state equation*

$$a_{k+1} = a_k - \mu\,(a_k - a_k^*) + .03\cos(\frac{2\pi k}{10})$$

which has both a time varying desired point, a_k^**, and a disturbance. We determine that the disturbance free equation*

$$a_{k+1} = a_k - \mu\,(a_k - a_k^*)$$

has a contractivity constant

$$\alpha = \max \frac{\|a_k - \mu\,(a_k - a_k^*) - a_k^*\|}{\|a_k - a_k^*\|} = 1 - \mu$$

For $\|\mu\| < 1$ *we have a contraction within any sized ball. Notice also that the linearity of the update allows this same contraction no matter how large or small our tube of interest,* $\mathcal{B}_{a_k^*}$*, is. Thus, the region of attraction includes the whole space. For a particular* a_k^**, for example*

$$a_k^* = \cos(\frac{2\pi k}{100})$$

We can bound the rate of time variation and use Theorem 3 to bound the convergent error. A plot showing the application of the bound in the theorem to this example with $\mu = .25$ *is provided in Figure 2.5. The top graph indicates the evolution of the actual trajectory,* a_k *in comparison to the desired trajectory* a_k^**, and shows the accuracy of the bound. The lower graph is a plot of the error,* $\|a_k - a_k^*\|$*, and the bound provided by the theorem. Notice that the bound is once again not particularly tight, because our assumptions restrict us to a worst case analysis in terms of the disturbance and time variation.*

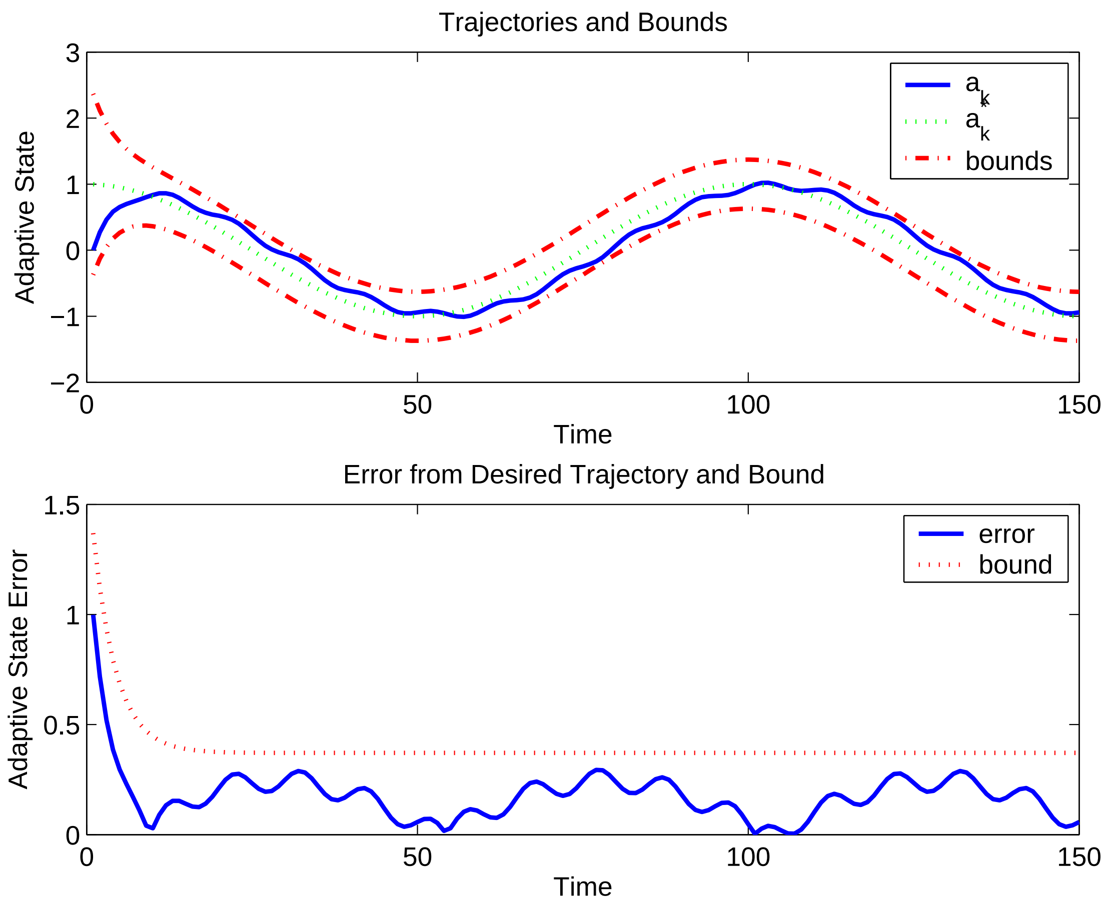

Figure 2.5: A sample application of theorem 3 indicating the robustness of exponential stability to disturbances and time variation.

2.2.4 Deterministic Averaging Theory

Using the previous theorems, we often assumed that an exact and accurate parameter change was available at every step of adaptation. Oftentimes, we are not so fortunate as to have such an accurate signal at every step. Usually, we can make up for this by having a signal which accurately indicates the correct adaptive state update on average over time. Many adaptive elements are derived with such an averaging mindset. In these elements, the adaptive state function does not give an update in exactly the correct direction, rather, an update that is in the correct direction only on average. Specifically, these algorithms usually have the form

$$\hat{a}_{k+1} = \hat{a}_k + \mu \hat{f}(k, \hat{a}_k, x_k) \tag{2.13}$$

where μ is some small parameter, $\mu \ll 1$, which controls the averaging. Since, for a particular input, we commonly are speaking about stability to a certain desired trajectory, a_k^*, we form the error system

$$\tilde{a}_{k+1} = \tilde{a}_k + \mu f(k, \tilde{a}_k, x_k) + a_k^* - a_{k+1}^* \tag{2.14}$$

where $\tilde{a}_k = \hat{a}_k - a_k^*$, $f(k, \tilde{a}_k, x_k) = \hat{f}(k, a_k^* + \tilde{a}_k, x_k)$, and μ is a small parameter upon whose size the averaging depends. We performed this change of variables because we want to study the boundedness of the error $\tilde{a}_k$ between our desired trajectory a_k^* and our actual trajectory $\hat{a}_k$. Deterministic averaging theory relates with (2.14) the following averaged system

$$\bar{a}_{k+1} = \bar{a}_k + \mu f_{av}(\bar{a}_k) + a_k^* - a_{k+1}^* \tag{2.15}$$

whose averaged update function is defined as

$$f_{av}(\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(i, \xi, x_i)$$

whose existence is one of our assumptions. Theorem 4 deals with the accuracy of the averaged approximation over a finite time period, $\{1, ..., \frac{T}{\mu}\}$. While Theorem 5 will deal with the accuracy of the approximation over an infinite amount of time.

Theorem 4 (Finite Time Deterministic Averaging: [3],[9],[10]). *Consider the adaptive element with adaptive state equation given by (2.13), error system given by (2.14), and averaged system given by (2.15). Assume that the desired trajectory changes slowly enough that*

$$\|a^*_{k+1} - a^*_k\| < c \qquad \forall k \in \{1, ..., T/\mu\}$$

We assume that μ is small enough such that the adaptive state error, $\|\tilde{a}_k\| = \|\hat{a}_k - a^*_k\|$, is less than h during this time period. Given that $\xi_k \in \mathcal{B}_0(h) = \{\xi_k | \|\xi_k\| < h\}$, we also assume that the adaptive state equation's update is bounded

$$\|f(k, \xi, x_k)\| \leq B_f \quad \forall \xi \in \mathcal{B}_0(h) \quad \forall k$$

Naturally, we will also need to characterize in some manner the adaptive state equation's deviation from its average. Since we are more likely to be able to characterize this behavior in a temporally averaged fashion, we introduce the following function

$$p(k, \xi) = \sum_{i=1}^{k} \left(f(i, \xi, x_i) - f_{av}(\xi)\right)$$

and assume that it has the following properties

$$\|p(k, \xi_1) - p(k, \xi_2)\| \leq L_p \|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2, \in \mathcal{B}_0(h)$$

$$\|p(k, a^*_k)\| \leq B_p \tag{2.16}$$

Furthermore, we assume that the averaged system is locally Lipschitz continuous

$$\|f_{av}(\xi_1) - f_{av}(\xi_2)\| \leq \lambda_f \|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2 \in \mathcal{B}_0(h)$$

and that the averaged system is uniformly contractive to the desired trajectory

$$\|\bar{a} + f_{av}(\bar{a})\| < \alpha\|\bar{a}\| \quad \forall \bar{a} \in \mathcal{B}_0(h)$$

Also assume that the time variation and the initial error are small enough such that

$$\|\bar{a}_1\| + \frac{c}{1-\alpha} + e^{\lambda_f T}\left(\|\bar{a}_1 - \tilde{a}_1\| + \mu(B_p + L_p h + L_p B_f T)\right) < h$$

If all of these assumptions are true, then we have

$$\|\tilde{a}_k - \bar{a}_k\| \leq e^{\lambda_f T}\left(\|\tilde{a}_1 - \bar{a}_1\| + \mu(B_p + L_p h + L_p B_f T)\right) \quad \forall k \in \{1, ..., T/\mu\}$$

$\diamond$ Defining the averaging error, $\Delta_k = \tilde{a}_k - \bar{a}_k$, we can begin by subtracting (2.14) and (2.15) to get

$$\Delta_{k+1} = \Delta_k + \mu\left(f(k, \tilde{a}_k, x_k) - f_{av}(\bar{a}_k)\right)$$

Running this recursion back in time gives

$$\Delta_{k+1} = \Delta_1 + \mu \sum_{i=1}^{k} \left(f(i, \tilde{a}_i, x_i) - f_{av}(\bar{a}_i)\right)$$

Adding a clever form of zero gives

$$\Delta_k = \Delta_1 + \mu \sum_{i=1}^{k} \left(f(i, \tilde{a}_i, x_i) - f_{av}(\tilde{a}_i) + f_{av}(\tilde{a}_i) - f_{av}(\bar{a}_i)\right) \tag{2.17}$$

Now, recalling our assumptions, we notice that we can write for $k \geq 2$

$$p(k, \xi) - p(k-1, \xi) = \sum_{i=1}^{k} \left(f(i, \xi, x_i) - f_{av}(\xi)\right) - \sum_{i=1}^{k-1} \left(f(i, \xi, x_i) - f_{av}(\xi)\right)$$

which will be true for any ξ. Subtracting the two sums shows that for $k \geq 2$

$$p(k, \tilde{a}_k) - p(k-1, \tilde{a}_k) = f(k, \tilde{a}_k, x_k) - f_{av}(\tilde{a}_k)$$

from which we can build the elements of (2.17)

$$\begin{aligned} \Delta_{k+1} = \; & \Delta_1 + \mu \textstyle\sum_{i=2}^{k} \left(p(i, \tilde{a}_i) - p(i-1, \tilde{a}_i) + f_{av}(\tilde{a}_i) - f_{av}(\bar{a}_i)\right) + \mu p(1, \tilde{a}_1) \\ & + \mu\left(f_{av}(\tilde{a}_1) - f_{av}(\bar{a}_1)\right) \end{aligned}$$

Rewriting the sums

$$\Delta_{k+1} = \Delta_1 + \mu \sum_{i=2}^{k} p(i, \tilde{a}_i) - \mu \sum_{i=2}^{k} p(i-1, \tilde{a}_i) + \mu \sum_{i=1}^{k} (f_{av}(\tilde{a}_i) - f_{av}(\bar{a}_i)) + \mu p(1, \tilde{a}_1)$$

and a change of indices of summation yields

$$\begin{aligned} \Delta_{k+1} = \quad & \Delta_1 + \mu \textstyle\sum_{i=2}^{k} p(i, \tilde{a}_i) - \mu \sum_{i=1}^{k-1} p(i, \tilde{a}_{i+1}) + \mu \sum_{i=1}^{k} (f_{av}(\tilde{a}_i) - f_{av}(\bar{a}_i)) \\ & + \mu p(1, \tilde{a}_1) - \mu p(1, \tilde{a}_2) \end{aligned}$$

Recombining the sums

$$\begin{aligned} \Delta_{k+1} \quad &= \Delta_1 + \mu \textstyle\sum_{i=2}^{k-1} (p(i, \tilde{a}_i) - p(i, \tilde{a}_{i+1})) + \mu p(k, \tilde{a}_k) \\ & \quad + \mu \textstyle\sum_{i=1}^{k} (f_{av}(\tilde{a}_i) - f_{av}(\bar{a}_i)) + \mu p(1, \tilde{a}_1) - \mu p(1, \tilde{a}_2) \end{aligned}$$

and using the triangle inequality and the Lipschitz continuity of the averaged system yields

$$\|\Delta_{k+1}\| \leq \|\Delta_1\| + \mu \sum_{i=1}^{k-1} \|p(i, \tilde{a}_{i+1}) - p(i, \tilde{a}_i)\| + \sum_{i=1}^{k} \lambda_f \|\tilde{a}_i - \bar{a}_i\| + \mu \|p(k, \tilde{a}_k)\|$$

Our assumption about the Lipschitz continuity of the total perturbation and its boundedness at zero error allows us to bound the last term in the sum with

$$\|p(k, \tilde{a}_k)\| \leq \|p(k, 0)\| + \|p(k, \tilde{a}_k) - p(k, 0)\| \leq B_p + L_p h$$

Substituting in the new bound gives

$$\|\Delta_{k+1}\| \leq \|\Delta_1\| + \mu \sum_{i=1}^{k-1} (L_p \|\tilde{a}_{i+1} - \tilde{a}_i\| + \lambda_f \|\tilde{a}_i - \bar{a}_i\|) + \mu(B_p + L_p h)$$

Using (2.14), we have

$$\|\Delta_{k+1}\| \leq \|\Delta_1\| + \mu \sum_{i=1}^{k-1} \left(L_p \|\mu f(i, \tilde{a}_i, x_i) + a_k^* - a_{k+1}^*\| + \lambda_f \|\tilde{a}_i - \bar{a}_i\|\right) + \mu(B_p + L_p h)$$

and our assumptions about the boundedness of f and the time variation give

$$\|\Delta_{k+1}\| \leq \|\Delta_1\| + \mu \sum_{i=1}^{k-1} (\mu L_p B_f + \lambda_f \|\Delta_i\| + c) + \mu(B_p + L_p h)$$

Rearranging

$$\|\Delta_{k+1}\| \leq \|\Delta_1\| + \mu^2 L_p B_f k + \mu c k + \mu\lambda_f \sum_{i=1}^{k-1} \|\Delta_i\| + \mu(B_p + L_p h)$$

and using the fact that we are dealing with a fixed amount of time, we have:

$$\|\Delta_{k+1}\| \leq \|\Delta_1\| + \mu(B_p + L_p h + L_p B_f T) + cT + \mu\lambda_f \sum_{i=1}^{k-1} \|\Delta_i\| \quad \forall k \in \{1, ..., T/\mu\}$$

Applying the discrete Bellman Gronwell identity[6], we have

$$\|\Delta_{k+1}\| \leq (1 + \mu\lambda_f)^k \left(\|\Delta_1\| + \mu(B_p + L_p h + L_p B_f T) + cT\right)$$

Recalling the Taylor series for the exponential, and that $e^{xk} \geq (1+x)^k \; \forall x \geq 0$, we can write

$$\|\Delta_{k+1}\| \leq e^{\mu\lambda_f k} \left(\|\Delta_1\| + \mu(B_p + L_p h + L_p B_f T) + cT\right)$$

Once again dealing with the fact that we are considering a finite amount of time we come to our final bound

$$\|\Delta_{k+1}\| \leq e^{\lambda_f T} \left(\|\Delta_1\| + \mu(B_p + L_p h + L_p B_f T) + cT\right) \quad \forall k \in \{1, ..., T/\mu\}$$

This indicates that if we start the averaged system and the unaveraged system at the same location, and if we require the time variation to be slow enough such that $c = O(\mu)$, we can adjust the maximum deviation of the unaveraged system from the averaged system using linear changes in μ. The last step in our argument is to verify that we never left the balls in which our assumptions were valid. Using the triangle inequality, we see that

$$\|\tilde{a}_k\| \leq \|\bar{a}_k\| + \|\Delta_k\|$$

[6] See appendix C.

and, since we had continuity of the averaged system

$$\|\bar{a}_{k+1}\| \leq \alpha\|\bar{a}_k\| + \|a^*_{k+1} - a^*_k\|$$

Running the recursion yields

$$\|\bar{a}_{k+1}\| \leq \alpha^k\|\bar{a}_0\| + \sum_{i=0}^{k-1} \alpha^{k-i}\|a^*_{i+1} - a^*_i\|$$

And the sum of an infinite geometric series yields

$$\|\bar{a}_{k+1}\| \leq \alpha^k\|\bar{a}_0\| + \frac{c}{1-\alpha}$$

Thus, if

$$\|\bar{a}_0\| + \frac{c}{1-\alpha} + e^{\lambda_f T}\left(\|\Delta_0\| + \mu(B_p + L_p h + L_p B_f T) + cT\right) < h$$

we never leave the ball within which we assumed we stayed. $\diamond$

We now consider an example application of the finite time deterministic averaging theorem that highlights the correctness and tightness of the bound. Notice that since we have included the information that our disturbance is not only bounded, but also has average zero, our bounds become tighter. On a quantitative level, this is not a feature of the theorem which matters much, since many of the constants needed to obtain such a bound will not be easily calculated in practice.[7] Rather, this theorem is better for its qualitative bounding information; namely, that as long as the time variation and the contraction constants are $O(\mu)$, you can make the averaging error over a finite time window arbitrarily small by shrinking the step size, μ, to an appropriate level. The requirement that α and c be $O(\mu)$, is a subtle

[7]Usually, we only know properties of the adaptive state functions, such as differentiability, which guarantee the existence of a Lipschitz constant. Even if we know the exact analytical form of the adaptive state function, finding Lipschitz constants analytically is typically fairly difficult.

nuance of the theorem, since our notation does not emphasize their dependence on μ.

Example 2.5 (Finite Time Averaging). *Consider the following adaptive state equation*

$$\tilde{a}_k = \tilde{a}_k - \mu\left(\tilde{a}_k - .2\right)^3 \cos\left(\frac{2\pi k}{10}\right)^2 + \mu \cos\left(\frac{2\pi k}{10}\right)$$

we see that we have an unaveraged update function

$$f(k, \tilde{a}_k) = -\left(\tilde{a}_k - .2\right)^3 \cos\left(\frac{2\pi k}{10}\right)^2 + \cos\left(\frac{2\pi k}{10}\right)$$

which has a corresponding averaged function

$$f_{av}(\tilde{a}_k) = \frac{-1}{2}\left(\tilde{a}_k - .2\right)^3$$

Consider the region of interest to be $[-.2, .6]$*, within this region, the averaged system has a Lipschitz constant*

$$\lambda_f = \max_{a \in [-.2,.6]} \left\| \frac{\partial f_{av}}{\partial a} \right\| = \max_{a \in [-.2,.6]} \left\| -\frac{3}{2}\left(a - .2\right)^2 \right\| = .24$$

Furthermore, the total perturbation

$$p(k, a) = \sum_{i=1}^{k} [f(i, a) - f_{av}(a)]$$

has a Lipschitz constant

$$\begin{aligned} L_p &= \max_{k \in \mathcal{N}, a \in [-.2,.6]} \left\| \textstyle\sum_{i=0}^{k} \frac{\partial f(i,a)}{\partial a} - \frac{\partial f_{av}}{\partial a} \right\| \\ &\leq \max_{k \in \mathcal{N}, a \in [-.2,.6]} \tfrac{3}{2}(a - .2)^2 \left\| \textstyle\sum_{i=1}^{k} \cos(\frac{4\pi i}{10}) \right\| + \left\| \textstyle\sum_{i=1}^{k} \cos(\frac{2\pi i}{10}) \right\| = 0.3142 \end{aligned}$$

and is bounded such that

$$p(k, a_k^*) \leq \max_k \left\| \sum_{i=1}^{k} \cos(\frac{2\pi i}{10}) \right\| = 2.1180 = B_p$$

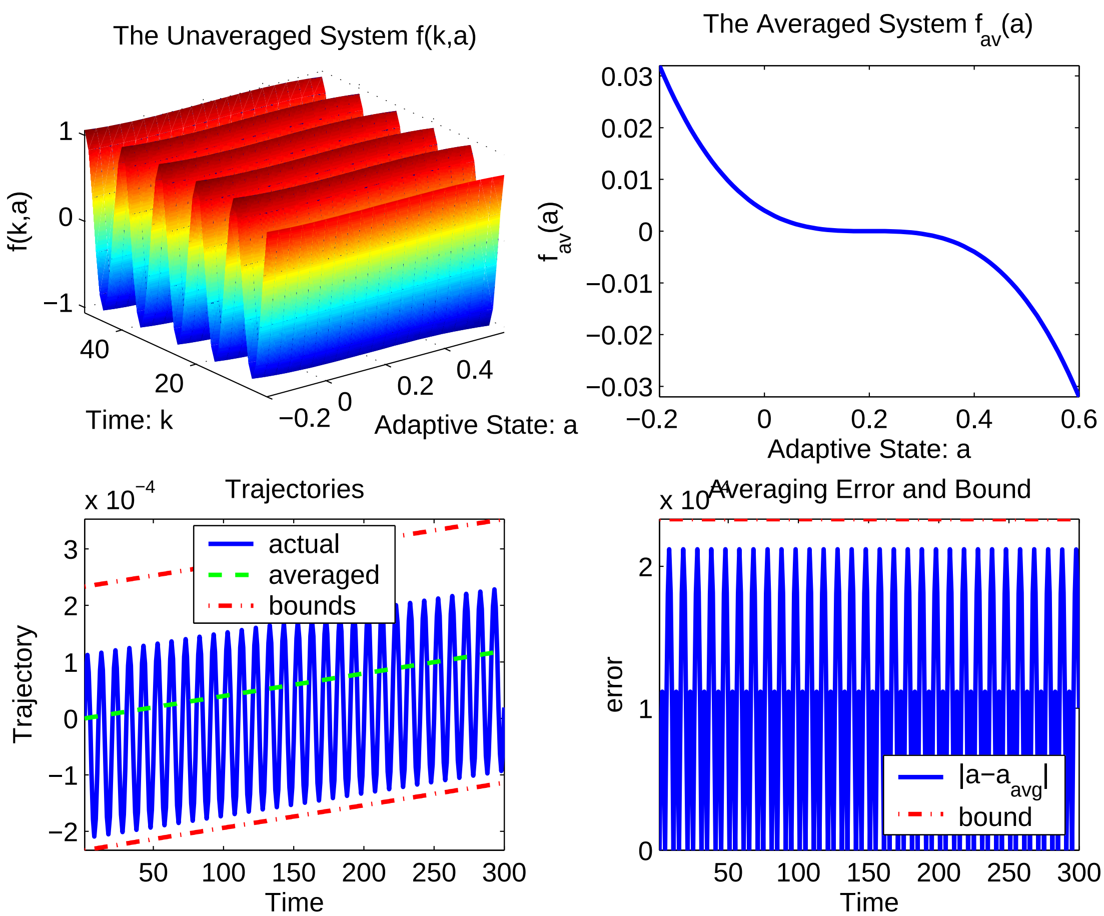

Figure 2.6: A sample application of the finite time averaging theorem.

Furthermore, we see that in our region of interest, the unaveraged function is bounded by

$$B_f = \max_{k,a\in[-.2,.6]} \left\| -(\tilde{a}_k - .2)^3 \cos(\frac{2\pi k}{10}) + \cos(\frac{2\pi k}{10}) \right\| < 1.640$$

We apply these numerical results in Figure 2.6 to indicate that the bound offered by Theorem 4 does indeed hold. For this figure, $\mu = .0001$.

Theorem 5 (Hovering Theorem: [10], [3]). *We now wish to extend our results from the previous section in order to bound the difference between the averaged and un-averaged systems over an infinitely long time window. To extend our results,* *though, we need to place a few more restrictions on the system. First of all, we require that the averaged system, (2.15), be Lipschitz continuous with a constant* α *less than one. Thus,*

$$\|\xi_1 + \mu f_{av}(\xi_1) - \xi_2 - \mu f_{av}(\xi_2)\| \leq \alpha \|\xi_1 - \xi_2\| \qquad \forall \xi_1, \xi_2 \in \mathcal{B}_0(h_a)$$

We also require that the fixed points of the map move sufficiently slowly, such that

$$\|a^*_{k+1} - a^*_k\| \leq c \quad \forall k$$

which will give us local exponential stability of the averaged algorithm. Furthermore, we define a windowed total perturbation

$$p_{nT/\mu}(k, \xi) = \sum_{i=nT/\mu}^{nT/\mu+k} [f(i, \xi, x_i) - f_{av}(\xi)]$$

which we assume is uniformly Lipschitz continuous and bounded $\forall k \in \{1, ..., T/\mu\}$ *such that*

$$\|p_{nT/\mu}(k, \xi_1) - p_{nT/\mu}(k, \xi_2)\| \leq \lambda_p \|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2 \in \mathcal{B}_0(h_a)$$

Then, if we require

$$h_a \geq \|\tilde{a}_1\| + \frac{c}{1-\alpha} + \frac{2 - \alpha^{T/\mu}}{1 - \alpha^{T/\mu}} \left(cT + \mu(B_p + L_p h + L_p B_f T)\right) e^{\lambda_f T} \qquad (2.18)$$

the difference between the averaged system's trajectory and the unaveraged system's trajectory is bounded by

$$\|\bar{a}_k - \tilde{a}_k\| \leq \frac{2 - \alpha^{T/\mu}}{1 - \alpha^{T/\mu}} \left(cT + \mu(B_p + L_p h + L_p B_f T)\right) e^{\lambda_f T}$$

Thus, the error from the desired trajectory in the unaveraged system is bounded by

$$\|\tilde{a}_{k+1}\| \leq \alpha^k \|\tilde{a}_1\| + \frac{c}{1-\alpha} + \frac{2 - \alpha^{T/\mu}}{1 - \alpha^{T/\mu}} \left(cT + \mu(B_p + L_p h + L_p B_f T)\right) e^{\lambda_f T}$$

$\diamond$ First of all, the conditions we placed on the averaged equation made it exponentially stable (see theorem 1), such that

$$\|\bar{a}_{k+1} - a_k^*\| \leq \alpha^k \|\bar{a}_0\| + \frac{c}{1-\alpha} \tag{2.19}$$

The concept for this proof, taken from [10], is to divide the time axis up into intervals of length T/μ, and let $\bar{a}_{nT/\mu,k}$, $k \in \{nT/\mu, nT/\mu+1, ..., (n+1)T/\mu\}$ be a solution to the averaged system on the nth interval. Furthermore, let $\bar{a}_{nT/\mu,nT/\mu} = a_nT/\mu$, so that at the begining of each interval, the interval averaged trajectories start with unaveraged system's value. Also, since the averaged system is exponentially stable, we have

$$\|\bar{a}_{nT/\mu} - \bar{a}_{(n-1)T/\mu,nT/\mu}\| \leq \alpha \|\bar{a}_{nT/\mu-1} - \bar{a}_{(n-1)T/\mu,nT/\mu-1}\|$$

Stepping back through time we have

$$\|\bar{a}_{nT/\mu} - \bar{a}_{(n-1)T/\mu,nT/\mu}\| \leq \alpha^{T/\mu} \|\bar{a}_{(n-1)T/\mu} - \bar{a}_{(n-1)T/\mu,(n-1)T/\mu}\|$$

And since we initialized the interval averaged trajectory with the state of the unaveraged error system, we have

$$\|\bar{a}_{nT/\mu} - \bar{a}_{(n-1)T/\mu,nT/\mu}\| \leq \alpha^{T/\mu} \|\bar{a}_{(n-1)T/\mu} - \tilde{a}_{(n-1)T/\mu}\|$$

Using the fact above and the triangle inequality gives

$$\|\bar{a}_{nT/\mu} - \tilde{a}_{nT/\mu}\| \leq \|\bar{a}_{nT/\mu} - \bar{a}_{(n-1)T/\mu,nT/\mu}\| + \|\bar{a}_{(n-1)T/\mu,nT/\mu} - \tilde{a}_{nT}\|$$

We recognize the term on the end as what we bounded with our theorem 4 (notice we have changed our requirements on the total perturbation ever so slightly so we can still use the finite time result). Replacing this term yields

$$\|\bar{a}_{nT/\mu} - \tilde{a}_{nT/\mu}\| \leq \alpha^{T/\mu} \|\bar{a}_{(n-1)T/\mu} - a_{(n-1)T/\mu}\| + (cT + \mu(B_p + L_p h + L_p B_f T))\, e^{\lambda_f T}$$

Running through the recursion in n and using the sum of an infinite geometric series gives

$$\|\bar{a}_{nT/\mu} - \tilde{a}_{nT/\mu}\| \leq \frac{(cT + \mu(B_p + L_p h + L_p B_f T))\, e^{\lambda_f T}}{1 - \alpha^{T/\mu}} \tag{2.20}$$

which bounds the error between the desired and actual trajectories at times nT/μ. Of course, we wish to bound this error at all times. Thus, we are interested in relating the bound on the error at these time instants to the error at other time instants. Without loss of generality choose n so that we can consider $k \in \{nT/\mu, nT/\mu + 1, \cdots, (n+1)T/\mu\}$.

$$\|\bar{a}_k - \tilde{a}_k\| \leq \|\bar{a}_k - \bar{a}_{nT/\mu,k}\| + \|\bar{a}_{nT/\mu,k} - \tilde{a}_k\| \tag{2.21}$$

Now, we note that the second term on the right hand side is bounded by Theorem 4. Thus, our attention turns the the first term on the right hand side. We have

$$\begin{aligned} \|\bar{a}_k - \bar{a}_{nT/\mu,k}\| &= \|\bar{a}_{k-1} + \mu f_{av}(\bar{a}_{k-1}) - \left(\bar{a}_{nT/\mu,k-1} + \mu f_{av}(\bar{a}_{nT/\mu,k-1})\right)\| \\ &\leq \alpha \|\bar{a}_{k-1} - \bar{a}_{nT/\mu,k-1}\| \end{aligned}$$

Reapplying this logic several times yields

$$\|\bar{a}_k - \bar{a}_{nT/\mu,k}\| = \alpha^{k-nT/\mu} \|\bar{a}_{nT/\mu} - \bar{a}_{nT/\mu,nT/\mu}\|$$

Because $\alpha < 1$, we see that over the interval $k \in \{nT/\mu, nT/\mu+1, \cdots, (n+1)T/\mu\}$ the largest the term on the right term will be for $k = nT/\mu$. Thus, we have

$$\|\bar{a}_k - \bar{a}_{nT/\mu,k}\| \leq \frac{(cT + \mu(B_p + L_p h + L_p B_f T))\, e^{\lambda_f T}}{1 - \alpha^{T/\mu}}$$

Returning to (2.21), we have

$$\begin{aligned} \|\bar{a}_k - \tilde{a}_k\| \leq \;& \tfrac{\left(cT + \mu(B_p + L_p h + L_p B_f T)\right) e^{\lambda_f T}}{1 - \alpha^{T/\mu}} \\ & + e^{\lambda_f T} \left(\mu(B_p + L_p h + L_p B_f T) + cT\right) \end{aligned}$$

which, after collecting terms, becomes

$$\|\bar{a}_k - \tilde{a}_k\| \leq \frac{2-\alpha^{T/\mu}}{1-\alpha^{T/\mu}} \left(cT + \mu(B_p + L_p h + L_p B_f T)\right) e^{\lambda_f T}$$

Using (2.20) and (2.19) via the triangle inequality gives

$$\|\tilde{a}_{k+1}\| \leq \alpha^k \|\tilde{a}_1\| + \frac{c}{1-\alpha} + \frac{2-\alpha^{T/\mu}}{1-\alpha^{T/\mu}} \left(cT + \mu(B_p + L_p h + L_p B_f T)\right) e^{\lambda_f T} \quad (2.22)$$

which is our desired bound. We must make sure we remained within the region that our assumptions were valid. For this to be true, we need

$$\tilde{a}_k \in \mathcal{B}_0(h_a) \Rightarrow \|\tilde{a}_k\| < h_a$$

which, using (2.22), can be guaranteed if

$$h_a \geq \|\tilde{a}_1\| + \frac{c}{1-\alpha} + \frac{2-\alpha^{T/\mu}}{1-\alpha^{T/\mu}} \left(cT + \mu(B_p + L_p h + L_p B_f T)\right) e^{\lambda_f T}$$

which is the condition, (2.18), required by the theorem. ⋄

As it was important for the finite time averaging theorem, it is also important to note that there is a hidden possible dependance of α and c on μ, and thus, one should stipulate conditions on this dependance if one wishes to draw qualitative conclusions about the implications of this theorem. As we have done in the past, we now include the following example to highlight the conditions and implications of the theorem.

Example 2.6 (Infinite Averaging with Time Variation). *Consider the system*

$$\hat{a}_{k+1} = \hat{a}_k - \mu \left(\hat{a}_k - .1\cos(\frac{2\pi k}{1000000}) \right) \cos(\frac{2\pi k}{10})^2 + \mu \cos(\frac{2\pi k}{10})$$

and its corresponding error system

$$\tilde{a}_k = \tilde{a}_k - \mu \tilde{a}_k \cos(\frac{2\pi k}{10})^2 + \mu\cos(\frac{2\pi k}{10}) + .1\cos(\frac{2\pi k}{1000000}) - .1\cos(\frac{2\pi(k+1)}{1000000})$$

and averaged system

$$\bar{a}_{k+1} = \bar{a}_k - \frac{\mu}{2}\bar{a}_k + .1\cos(\frac{2\pi k}{1000000}) - .1\cos(\frac{2\pi(k+1)}{1000000})$$

Define the region of interest to be $[-.5, .5]$*. We can then deduce that* $\lambda_f = 1/2$*, and*

$$L_p = \max_k \frac{1}{2}\|\sum_{i=1}^{k}\cos(\frac{4\pi i}{10})\|$$

Furthermore, we note that $B_f = 1.5$*. Figure 2.7 shows the bound that Theorem 5 provides for this example, optimized over* T *to make the bound on the averaging for the chosen* μ *be as small as possible. We see that the unaveraged system follows the averaged system closely for this* μ*, and that both of them track the time-variation. It is important to note that the time variation is very slow in this example, so that the* μ *can be small enough to make the averaging error tiny, and yet still large enough to track the time variations.*

The Hovering theorem is a powerful theorem to use for adaptive elements. It indicates that, under a few sufficient conditions, we can design adaptive elements by using state equations which update in the correct direction only on average. This is, in fact, the way that most adaptive devices are designed, since exact solutions for the optimal settings for future inputs may be impossible to determine from a small amount of noisy current data. Instead, we typically find an "error signal" which only on average indicates the inaccuracy of our current parameter settings and use this to adapt our system. It should be noted that this theorem, like Theorem 8 is rarely applied in a quantitative manner, as we have done in the example, because in most applications it is easy to prove the Lipschitz constants exist, but they are difficult to calculate. However, this does not matter too much, since the major implications of the theorem are qualitative ones. Specifically, the

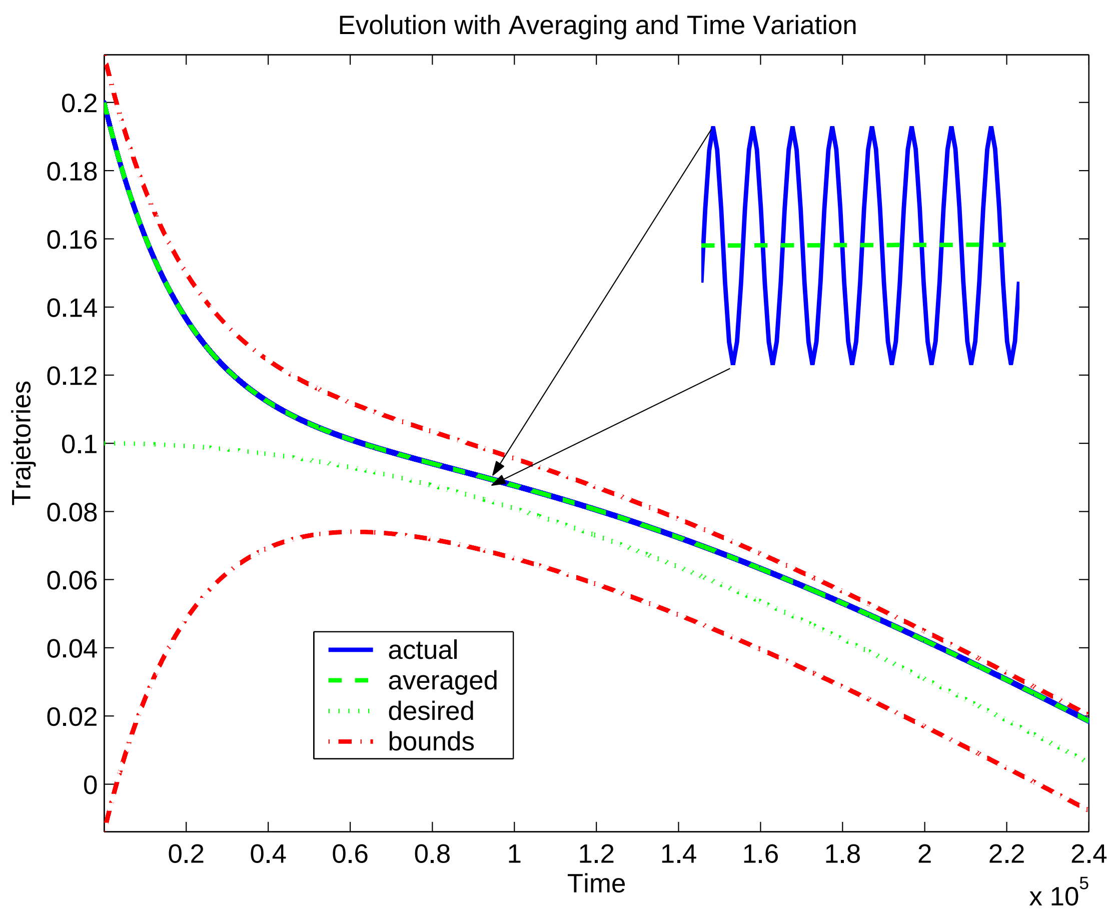

Figure 2.7: An example application of the hovering theorem, showing averaging applied to a time varying system. The unaveraged system hovers closely around the averaged system, which is tracking the time variation, and we see that the bounds are accurate, and fairly tight towards the end of this example.

important meaning of the theorem is that if we design an adaptive element with a state equation which is contractive only on average to a stationary point of interest, for a class of inputs such that the contraction constant and time variation are $O(\mu)$, we can make the adaptive element behave close to its averaged behavior by shrinking the step size.

2.3 Examples of Adaptive Devices Found in Digital Receivers

One arena in modern engineering that has reaped many of the benefits of simple adaptive devices lies within the physical layer of digital receivers. We include here many examples of adaptive devices found in the single carrier regime of digital receiver types. The inclusion is motivated by our discussion in later chapters about the interaction between different types of adaptive algorithms found in digital receivers. This arena seems to be one filled with interesting possible applications of a theory of interactions between adaptive devices, and, since we are interested in establishing the basis of such a theory, we will include a wide range of fundamental algorithms, including those we wish to study here. Because we will be interested in the interactions between different receiver subsystems, for many of the adaptive elements we list the averaged adaptive state functions[8] which include parameters from other adaptive elements. Since receiver devices are typically analyzed under the assumptions that other devices are behaving in an ideal manner, the calculation of these sensitivity functions has often not been carried out in the past. Hence, we include our derivations for the sensitivity functions we list in Tables 2.1 through 2.4 in Appendix A.

[8] We shall call these "sensitivity functions" from now on.

To classify the inputs and algorithms of interest to us, we begin by deriving a model for a received PAM signal subjected to a channel with time-varying multipath distortion and additive noise. In a typical PAM transmission system, the signal that the transmitter transmits is

$$s(t) = \mathrm{Re}\left\{ e^{j\left(2\pi f_{c_t} t + \theta_t(t)\right)} \sum_k a_k p(t - kT) \right\} \tag{2.23}$$

The received signal at passband is

$$u(t) = \sum_{i=1}^{Q} c_i(t) s(t - \tau_i(t)) + n(t) \tag{2.24}$$

After rough analog downconversion we have

$$r(t) = \mathrm{lpf}\left\{ e^{-j(2\pi f_{c_r} t + \theta_r(t))} u(t) \right\} \tag{2.25}$$

Assuming the lpf passes the signal component and attenuates the noise,

$$r(t) = \sum_k a_k \sum_{i=1}^{Q} c_i(t) e^{j\left(2\pi(f_{c_t} - f_{c_r})t + 2\pi f_{c_t}\tau_i(t) + \theta_t(t - \tau_i(t)) - \theta_r(t)\right)} p(t - \tau_i - kT) + \tilde{n}(t) \tag{2.26}$$

where $\tilde{n}(t) = \mathrm{lpf}\left\{ n(t) e^{j2\pi f_{c_r} t + \theta_r(t)} \right\}$. Defining $f_c = f_{c_t} - f_{c_r}$ and $\theta(t) = 2\pi f_{c_t} \tau_i(t) + \theta_t(t - \tau_i(t)) - \theta_r(t)$ for compactness, we have

$$r(t) = \sum_k a_k \sum_{i=1}^{Q} c_i(t) e^{j(2\pi f_c t + \theta(t))} p(t - \tau_i - kT) + \tilde{n}(t) \tag{2.27}$$

After sampling the signal, we have

$$r_m = \sum_k a_k \sum_{i=1}^{Q} c_i(mT_s) e^{j(2\pi f_c T_s m + \theta(mT_s))} p((m - k)T - \tau_i) + \tilde{n}_m \tag{2.28}$$

which is the digital received signal that the receiver must process in such a way as to recover the original transmitted symbols, a_k.

2.3.1 Automatic Gain Control

In order to keep the dynamic range of the digital quantization fully used, digital receivers often use a gain control to keep the power at the input analog to digital convertor to be roughly a constant. These gain controls can be thought of as adaptive devices, and one gradient update we might consider, [11], for their operation is

$$g_{k+1} = g_k - \mu_G \left(g_k^2 x_k^2 - d \right) \operatorname{sign}[g_k] \tag{2.29}$$

2.3.2 Carrier Recovery

Due to the unknown delay between the transmitter and the receiver, and the phase difference between their carrier oscillators, receivers often employ some sort of adaptive carrier synchronization. The phase estimates can be formed in a number of adaptive ways (see, e.g., [11] or [12]), and some possible error functions for a recursive adaptive estimator are shown in Table 2.1.

2.3.3 Timing Recovery

The unknown delay between the transmitter and the receiver along with the oscillator inaccuracies also induces a baud timing offset. Digital receivers often use an adaptive algorithm to recursively estimate and track this timing offset [11], [12]. A wide variety of timing algorithms have been proposed over the years, and a few of them are listed in Table 2.2.

Table 2.1: Error Detectors for Carrier Phase Recovery and their Sensitivities

Name	Update	Sensitivity
PLL [12]	$\mathrm{Im}\left\{\hat{a}_k^* x_k e^{-j\hat{\theta}_k}\right\}$	$g_k \mathsf{E}[\lvert a\rvert^2] \lvert h_{\hat{\tau},\mathbf{f}}[k,k]\rvert \sin\left(\angle h_{\hat{\tau},\mathbf{f}}[k,k] - \hat{\theta}\right)$
Low SNR ML BPSK [12]	$\mathrm{Re}\left\{x_k e^{-j\hat{\theta}_k}\right\} \mathrm{Im}\left\{x_k e^{-j\hat{\theta}_k}\right\}$	
Low SNR ML M-PSK [12]	$-\sum_{m=0}^{M/2-1}\left[\mathrm{Re}\left\{x_k e^{-j\hat{\theta}_k}\right\}\right]^3 \left[\mathrm{Im}\left\{x_k e^{-j\hat{\theta}_k}\right\}\right]$	
Fourth Power [13]	$\mathrm{Im}\left\{\left(x_k e^{-j\hat{\theta}_k}\right)^4\right\}$	$\lvert \mathsf{E}\left[a^4\right] \sum_n h_{\hat{\tau},\mathbf{f}}^4[k,n]\rvert \sin\left(\angle \sum_n h_{\hat{\tau},\mathbf{f}}^4[k,n] + \angle \mathsf{E}\left[a^4\right] - \hat{\theta}_k\right)$

2.3.4 Equalization

In digital radio systems, (possibly time varying) multipath propagation linearly distorts the received signal. In order to mitigate the effects of multipath, and still have low complexity, digital receivers often employ linear equalizers. These equalizers are one of the most familiar forms of an adaptive signal processing algorithm. A list of a few adaptive equalization algorithms is shown in Table 2.4.

Table 2.2: Timing Error Detectors for Bandpass Signals.

Name	**Update**
ML-Based [12]	$\text{Re}\left\{\hat{a}_k^* \left(y'(kT+\hat{\tau}_k)e^{-j\hat{\theta}_k} - \sum_{m=k-D}^{k+D} \hat{a}_m h'\left((k-m)T\right)\right)\right\}$
Zero Crossing [12]	$\text{Re}\left\{\left(\hat{a}_{k-1}^* - \hat{a}_k^*\right) y(kT - T/2 + \hat{\tau}_{k-1})e^{-j\hat{\theta}_k}\right\}$
Early Late [12]	$\text{Re}\left\{\hat{a}_k^* e^{-j\hat{\theta}_k}\left[y(kT+T/2+\hat{\tau}_k) - y(kT-T/2+\hat{\tau}_{k-1})\right]\right\}$
Muell. & Muell. [14]	$\text{Re}\left\{\hat{a}_{k-1}^* y(kT+\hat{\tau}_k)e^{-j\hat{\theta}_k} - \hat{a}_k^* y((k-1)T+\hat{\tau}_{k-1})e^{-j\hat{\theta}_k}\right\}$
M-Nonlin. [12]	$\lvert y(kT+\hat{\tau}_k)\rvert^{M-2}\text{Re}\left\{y^*(kT+\hat{\tau}_k)y'(kT+\hat{\tau}_k)\right\}$
M-Nonlin. Numer. [11]	$\lvert y(kT+\hat{\tau}_k)\rvert^{M-1}\left(\lvert y(kT+\hat{\tau}_k+\delta)\rvert - \lvert y(kT+\hat{\tau}_k-\delta)\rvert\right)$
NDA-Early Late [12]	$\text{Re}\left\{y^*(kT+\hat{\tau}_k)\left[y(kT+T/2+\hat{\tau}_k) - y(kT-T/2+\hat{\tau}_{k-1})\right]\right\}$
Gardner [15]	$\text{Re}\left\{\left[y(kT-T+\hat{\tau}_{k-1}) - y(kT+\hat{\tau}_k)\right]y^*(kT-T/2+\hat{\tau}_{k-1})\right\}$
Bandedge [16]	$\text{bpf}_{f_c-\frac{1}{2T}}[y](kT+\hat{\tau}_k)^*\text{bpf}_{f_c+\frac{1}{2T}}[y](kT+\hat{\tau}_k)$

Table 2.3: Sensitivities of Timing Error Detectors for Bandpass Signals.

Name	**Sensitivity**
Zero Crossing [12]	$\mathrm{Re}\left\{\begin{array}{l} \hat{g}_k e^{-j\hat{\theta}_k} \mathsf{E}\left[\vert a\vert^2\right] \sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-l-\frac{1}{2})T+\hat{\tau}_{k-1}) \\ \left[p((\frac{1}{2}-l)T+\hat{\tau}_{k-1}-\tau_i) - p((-\frac{1}{2}-l)T+\hat{\tau}_{k-1}-\tau_i)\right] f_{l,k} \end{array}\right\}$
Early Late [12]	$\mathrm{Re}\left\{\begin{array}{c} \hat{g}_k e^{-j\hat{\theta}_k} \mathsf{E}[\vert a\vert^2] \Big(\sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-l+\frac{1}{2})T+\hat{\tau}_k) p((\frac{1}{2}-l)T+\hat{\tau}_k-\tau_i) f_{l,k} \\ -\sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-l-\frac{1}{2})T+\hat{\tau}_{k-1}) p((-\frac{1}{2}-l)T+\hat{\tau}_{k-1}-\tau_i) f_{l,k}\Big) \end{array}\right\}$
Muell. & Muell. [14]	$\mathrm{Re}\left\{\begin{array}{c} \hat{g}_k e^{-j\hat{\theta}_k} \mathsf{E}[\vert a\vert^2] \sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-l)T+\hat{\tau}_k) p((1-l)T+\hat{\tau}_k-\tau_i) f_{l,k} \\ -\hat{g}_{k-1} e^{-j\hat{\theta}_{k-1}} \mathsf{E}[\vert a\vert^2] \sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-1-l)T+\hat{\tau}_{k-1}) p((-1-l)T+\hat{\tau}_{k-1}-\tau_i) f_{l,k-1} \end{array}\right\}$
2-Nonlin. [11]	$\vert g_k\vert^2 \mathsf{E}[\vert a_n\vert^2] \sum_n \mathrm{Re}\left\{h^*_{\tau,\mathbf{f}}[k,n] dh_{\tau,\mathbf{f}}[k,n]\right\} + \sigma^2$

Table 2.4: Cost Functions and Update Terms for Various Equalizers.

Name	**Cost Function**	**Update**
LMS [11]	$\mathsf{E}\left[\lvert d_{k-\delta} - \mathbf{r}_k\mathbf{f}\rvert^2\right]$	$\left(\hat{d}^*_{k-\delta} - \mathbf{r}_k^T\mathbf{f}\right)\mathbf{r}_k^*$
Zero Forcing [17]		$(d_{k-\delta} - \mathbf{r}_k\mathbf{f})d^*_{k-m-\delta}$
Sato [18]		$\left(\mathbf{r}_k^T\mathbf{f} - \alpha\mathrm{csgn}(\mathbf{r}_k^T\mathbf{f})\right)\mathbf{r}_k^*$
CMA [19], [20]	$\mathsf{E}\left[(\gamma - \lvert\mathbf{r}_k\mathbf{f}\rvert^p)^2\right]$	$\lvert\mathbf{r}_k^T\mathbf{f}\rvert^{p-2}\left(\gamma - \lvert\mathbf{r}_k^T\mathbf{f}\rvert^p\right)\mathbf{r}_k^T\mathbf{f}\mathbf{r}_k^*$
BGR [21]		$\left(\beta_1(d_k - \mathbf{r}_k^T\mathbf{f}) + \beta_2(\mathbf{r}_k^T\mathbf{f} - \alpha\mathrm{csgn}(\mathbf{r}_k^T\mathbf{f})\right)\mathbf{r}_k^*$

Table 2.5: Sensitivities for Various Equalizers.

Name	**Sensitivity Function**
LMS [11]	$\hat{g}_{k-m}e^{j\hat{\theta}_{k-m}}\mathsf{E}[\lvert a\rvert^2]h^*_{\hat{\tau}}[k-m,k-\delta] - \sum_{l=0}^{P-1}\hat{g}_{k-m}\hat{g}_{k-l}$ $e^{j(\hat{\theta}_{k-m}-\hat{\theta}_{k-l})}\sum_n \mathsf{E}[\lvert a\rvert^2]h^*_{\hat{\tau}}[k-m,n]h_{\hat{\tau}}[k-l,n]f_l + \sum_{l=0}^{P-1}\mathsf{E}[v^*_{k-m}v_{k-l}]f_l$
Zero Forcing [17]	$\mathsf{E}[\lvert a\rvert^2]\left(\delta[m]-\sum_{i=1}^{N} h[k-i,k-m-\delta]\right)$
Godard/CMA [19], [20]	$\gamma\begin{bmatrix} \mathsf{E}[\lvert a\rvert^2]\sum_n\sum_i \\ h^*[k-i,n] \\ h[k-i,n]f_i \\ +f_m\mathsf{E}[\lvert v_{k-m}\rvert^2] \end{bmatrix} - \begin{bmatrix} \sum_{n_1}\sum_{n_2}(\delta[n_1-n_2]\mathsf{E}[\lvert a\rvert^4]+(1-\delta[n_1-n_2])(\mathsf{E}[\lvert a\rvert^2])^2) \\ \sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}h[k-i_1,n_1]h^*[k-i_3,n_1]h[k-i_2,n_2]h^*[k-m,n_2] \\ +2\sum_{n_1}\mathsf{E}[\lvert a\rvert^2]\sum_{i_1}\sum_{i_2}\lvert f_{i_1}\rvert^2 f_{i_2}\mathsf{E}[\lvert v\rvert^2]h[k-i_2,n_1]h^*[k-m,n_1] \\ +2\sum_n \mathsf{E}[\lvert a\rvert^2]f_{k-m}\mathsf{E}[\lvert v_{k-m}\rvert^2]\sum_{i_2}\sum_{i_3} f_{i_2}f^*_{i_3}h[k-i_2,n]h[k-i_3,n] \\ +\sum_i \lvert f_i\rvert^2 f_m(\delta[i-m]\mathsf{E}[\lvert v_i\rvert^4]+(1-\delta[i-m])(\mathsf{E}[\lvert v_i\rvert^2])^2) \end{bmatrix}$

Chapter 3

Binary Adaptive Compounds: Combining and Configuring Two Adaptive Blocks

In this chapter, we consider the possibility of connecting two adaptive elements together into a binary adaptive compound (BAC) from an engineering mindset. We explore possible reasons for doing such a thing, as well as discuss practical methods which lead to the design of distributed adaptive compounds. Then, we consider some possible ways to connect two adaptive signal processing elements together, and select one - a series connection - to focus on for the rest of the thesis.

3.1 Why might we combine two adaptive elements?

Oftentimes in engineering we are faced with problems that require adaptivity, but can not be solved with only one adaptive element. There are a multitude of situations which dictate a distributed adaptive structure. Sometimes distributed adaptation is given to us by nature. As we discussed in the introduction, the economy and our own brains can be view as distributed adaptive systems. In the case of the economy, the different adaptive elements are the businesses and individuals who interact to make the larger economic system. Our brains could be viewed as adaptive systems, where the separation of the adaptive elements indicated separation of different parts or functions of the brain.

Returning to an engineering perspective, perhaps the most common factor inviting distributed adaptation occurs when implementation simplicity or physical constraints, such as the physical separation of a transmitter and a receiver, dictate a distributed structure. In such a situation not all of the transmitter parameters may be known to the receiver, and the transmitter and receiver sit on opposite sides of a channel with uncertainty, forcing each side to be possibly engineered to infer something about the other. Another possible reason for using distributed adaptation occurs when we have multiple optimizations that we wish to perform that depend on how well the other optimizations are doing. One way to perform these optimizations is to come up with a scheme where separate adaptive elements work together to assure that the different optimizations are achieved. In such situations, for example power control for cellular telephones, a distributed structure may perform this optimization more efficiently that a situation with a central authority controlling the optimization. Specific instances of distributed adaptation occur in digital receivers, in traffic control, and in neural processing, to name a few. In still other cases, the multiple adaptation is a relic of our method of solution of the overall design problem. That is, we went about making the design by dividing the overall problem up into sub-problems, and then designing an adaptive element for each problem. This is a technique that can be likened to a problem solving strategy known as "divide and conquer," and we wish to characterize it and express what it means for the engineering of adaptive systems.

3.1.1 The Divide and Conquer Mindset: General Description

Design engineers are often faced with gargantuan tasks. Today's electronic and computing devices have become so complex that it is hard to imagine how such a complex device was ever even planned, let alone implemented. One technique that practicing engineers, as well as practical people, use to tackle complex tasks is the divide and conquer method. A very difficult task is split into a number of sub-tasks, which may be easier to handle if dealt with independently of the other sub-tasks. In engineering problems, this often translates into the assumption that all of the other sub-tasks have been solved perfectly when designing the solution to a particular sub-task. In the context of adaptive systems, we define the notion of design by divide and conquer to mean the selection of several interconnected adaptive elements performing different tasks, each of which we guarantee to operate correctly, if all of the other devices upon which they depend are operating perfectly. Naturally, a cautious engineer wishes to determine whether or not such a strategy leads to devices that work properly after connecting the adaptive elements together especially if none are initialized at their perfect operating settings. Evidence that such a study is a worthwhile effort is provided in Appendix B where we have taken excerpts from the digital receiver literature which discuss the interaction of adaptive receiver components which have been designed independently of one another. We can see from these quotes that, while recognition of the interaction of the adaptive elements is cited as a problem, there is virtually no basic general theory which is used to discuss and deal with this interaction. Each specific interconnection of adaptive elements found in a receiver has its own associated folklore of design tips and warnings. We will begin to fill this void by characterizing the behavior of interconnected adaptive elements within a stability theory context for a particular two element case in this thesis, but, first, we need to consider the possible ways of interconnecting the adaptive elements designed by divide and conquer.

3.2 How might we combine two Adaptive Elements?

There are quite a few ways we can imagine we might hook two adaptive signal processing elements together. Restricting our attention to combinations that process single (or single bundles of) input signals to create a single (or single bundles of) output signals we can come up with at least four ways to combine two adaptive elements, which are shown in Figures 3.1, 3.2, 3.3, and 3.4.

One possibility is to process the signal input into the compound first by one element, then process the output of the first element with the second element, as shown in Figure 3.1. We refer to this connection as a *Series Feed-forward Binary Adaptive Compound*, or SFFBAC. This is perhaps one of the simplest ways of hooking the two devices together one can imagine, and, other than to note the possibility and existence of some other structures, we focus entirely upon this SFFBAC configuration for the rest of the thesis.

Another possibility is to still feed the output of the first element into the input of the second element, but to also allow the first element to adapt from the second element's output. This configuration is referred to as a *Series Feed-back Binary Adaptive Compound*, and can be see in Figure 3.2. This configuration can be commonly found, among elsewhere, in digital QAM communications receivers, where, for example, the carrier recovery loop can sit inside the adaptive equalizer's adjustment loop. While we will not focus on this type of structure in the current thesis, we hope to include it as part of a future research effort.

Forgoing a serial processing based structure, we could consider structures in which the input signal is processed separately in parallel by both adaptive elements, and then their outputs in combined in some sort of meaningful way. One possibility is then to adapt each of the two adaptive elements only based on their own inputs,

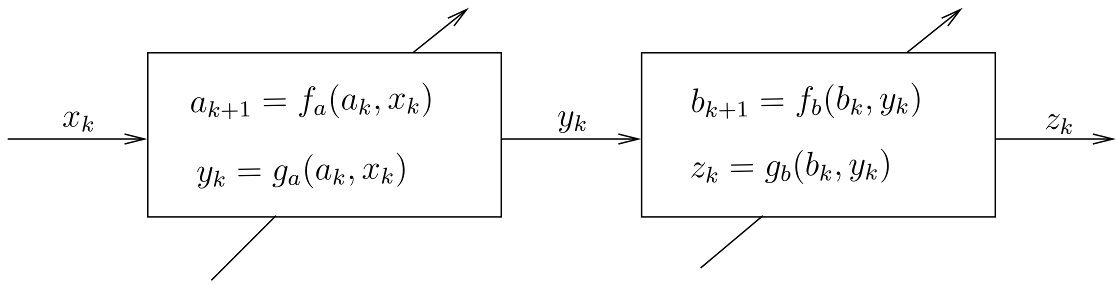

Figure 3.1: Two adaptive devices connected in series feed-forward form.

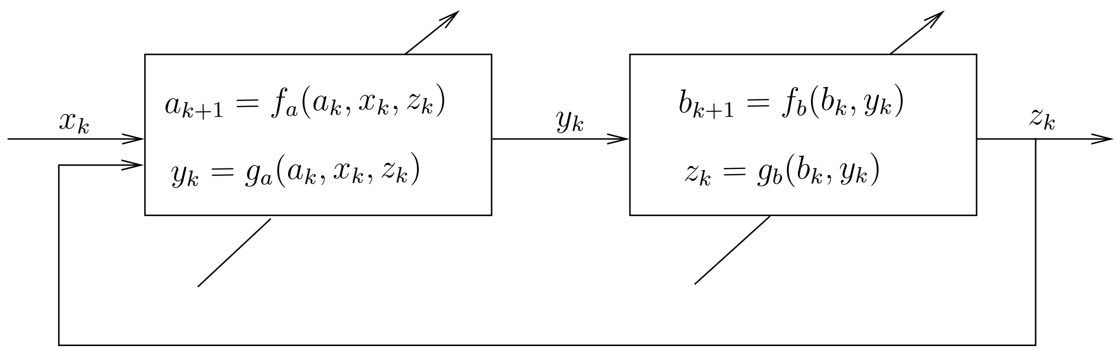

Figure 3.2: Two adaptive devices connected in series feed-back form.

and we refer to such a structure as a *Parallel Combining Feed-forward Binary Adaptive Compound*. Other than to mention the possibility of connecting two adaptive devices in this way, and to provide Figure 3.3, we do not provide any attention to this possibility in this thesis. Exploring its operation is a possibility to be mentioned in the Future Work section.

The last way mentioned here of connecting two adaptive elements together to process one signal that we consider is diagrammed in Figure 3.4. In this situation, we adapt the two elements independently on the same input signal, but also allow each of the elements to use the output of the combiner to adapt as well. Because it feeds-back the output signal, we refer to this structure as a *Parallel Combining Feed-back Binary Adaptive Compound*. It too is a possible avenue for further research.

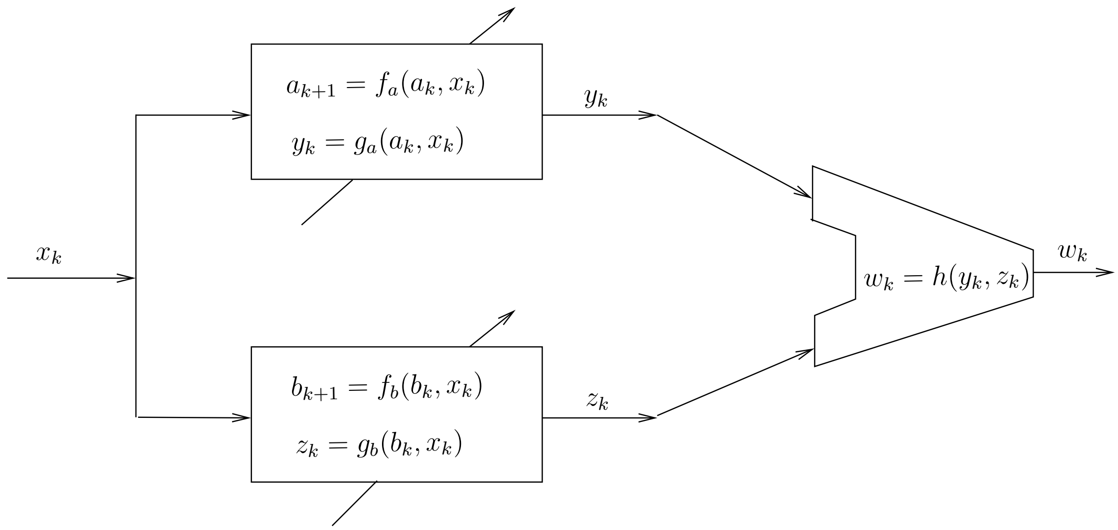

Figure 3.3: Two adaptive devices connected in parallel combining feed-forward form.

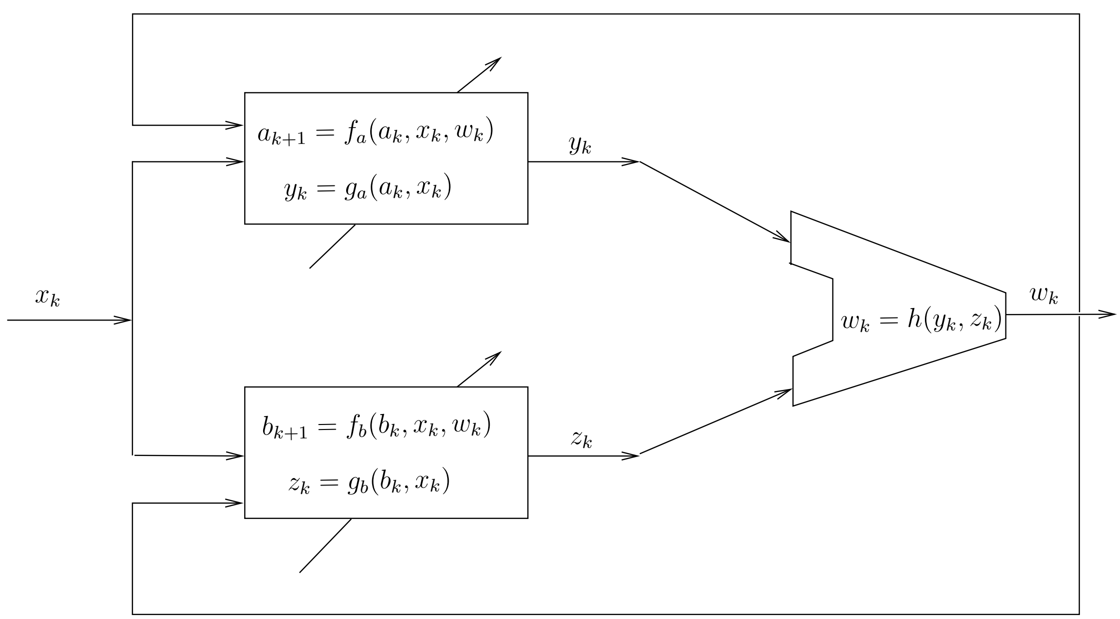

Figure 3.4: Two adaptive devices connected in parallel combining feed-back form.

3.3 Divide and Conquer: What does it mean for different binary structures?

Continuing our discussion of binary adaptive compounds from an engineering mindset, the next logical step after considering the different physical ways to interconnect the two adaptive elements, and selecting one for our focus, is to consider whether or not such a connection will create a device which processes the input to the binary adaptive compound in the desired way. In this section, we introduce a set of specific guidelines for designing series feedforward binary adaptive compounds which we will call the divide and conquer conditions. Note that a significant further narrowing of our focus has occurred here, since we could consider at least four ways to connect the two adaptive elements into a BAC, and now we are only investigating one. With this narrowing of focus, however, comes the ability to increase our mathematical rigor. In the next chapter we will develop a rigorous theory that show that the DaC conditions given for SFFBACs can lead to systems that "work", where we continue to specify what it means for a system to work within the context of dynamical systems and stability and boundedness to desired state and output trajectories. When we are done, we will have a set of conditions and a handful of theorems which we can use to guide study and design of SFFBACs, as well as larger feedforward adaptive systems.

3.3.1 Combined performance of desired points

Recall that when we characterized the behavior of an adaptive element in Chapter 2, we discussed the behavior of adaptive signal processing elements. In that chapter, we stipulated that, given a particular input, we design the adaptive subsystem to be locally stable (or uniformly bounded) relative to a point in the adaptive state space which, when given together with the input to the processing system, produced a desirable output. Similarly, when we combine two adaptive elements into a binary adaptive compound, we are interested in them both being locally stable to (or uniformly bounded relative to) points in their state spaces which produce an overall desired output. Thus, the very first item to consider when connecting two adaptive elements together is the overall performance (if such a measure exists) of the output of the system when the two adaptive elements are at their "desired" points. If the desired points do not correspond to a desired overall output, then either the algorithms must be modified, or different algorithms must be chosen. Thus, we begin our discussion here by assuming that two adaptive elements have been chosen that have "desired" points such that when fixed at the "desired" points and hooked together in the fashion (i.e. SFFBAC) that we are investigating, they produce a desired output. Once this is true, given a continuous performance measure for the overall system, the remaining questions involve stability and boundedness of the combined system to the desired points, and these are the questions which we will address in the next chapter. First, we collect the idea just mentioned with other rules for connecting two adaptive elements into a SFFBAC in a list which offers qualitative conditions to check when designing a SFFBAC.

3.3.2 What does DaC mean for Series Feedforward Structures?

We now introduce qualitative sufficient conditions for designing a series feedforward binary adaptive compound using two independently designed adaptive elements. While these conditions are not always necessary, their sufficiency guarantees that, as long as we obey them, our SFFBAC will behave properly. We will prove this quantitatively and rigorously in the next chapter, but first, we list the qualitative conditions.

The Divide and Conquer Conditions. Consider two adaptive elements to be connected together to form a Series Feed-forward binary adaptive compound (SFFBAC), as shown in Figure 3.1. For the possible inputs of interest, we require the following conditions.

Condition 1 (Optimal Desired Trajectories). *If their adaptive state is locked onto their desired trajectory, the two adaptive elements produce an overall output which is desirable.*

Condition 2 (Continuous Performance). *The system has an overall performance measure, whose value will be used to judge how well the entire system is performing. If such a measure exists, the adaptive compound should be continuous with respect to the adaptive state parameters of the two elements.*

Condition 3 (Contractivity of First Element). *The adaptive element to appear first in the SFFBAC must be (at least on average), locally contractive to the desired trajectory.*

Condition 4 (Contractivity of Second Element). *The adaptive element to appear second in the SFFBAC must be (at least on average), locally contractive to its desired trajectory when the first element is fixed at its desired trajectory.*

Condition 5 (Sufficient Continuity). *The second element's adaptive state equation is (at least on average) continuous with respect to the first elements adaptive state coupled through the first elements input-output equation.*

Condition 6 (Small Perturbations). *The time variations must be slow enough, the disturbances small enough, the step sizes (if the elements are of the averaging form) small enough, and the initializations accurate enough (ie close enough to the desired points).*

We will prove in the next chapter that, if these sufficient conditions are satisfied, then the SFFBAC will be locally exponentially stable to a ball of finite size surrounding the desired points. This local stability implies that, if the performance is Lipschitz continuous with respect to the two adaptive elements' states, then we will be "close" to the desired performance. That being said, we drop the notion of an overall system performance from now on, since asking for performance close to the desired performance (=performance at stationary points), is the same as asking for adaptive states close to the desired adaptive states for the cases which the DaC conditions consider. Thus, from this point on in the thesis, it is assumed that Conditions 2 and 1 are true. The remainder of this thesis is dedicated to first making the remaining conditions quantitatively rigorous, and then applying them to an example SFFBAC found in communications systems.

Chapter 4

Designing Series Feedforward DASPs with DaC

In the previous chapter, we outlined a set of qualitative guidelines for connecting two adaptive elements together to form a SFFBAC. In this chapter, we set about making these guidelines mathematically concrete by providing in sections 4.1.1-4.1.4 the necessary theorems for local exponential stability and boundedness for our adaptive elements. We include examples after each theorem to indicate their quantitative accuracy, as well as highlight their qualitative importance.

After we are done showing how to make an SFFBAC behave well, we provide section 4.2, which gives conditions under which we can expect it to misbehave by not tracking the desired states. This can be of use when designing a SFFBAC, because it indicates the possible failure modes of the system.

The material in this chapter will provide the theoretical basis for the material in Chapter 5, which examines some examples of SFFBACs found in communications systems, and we show how the theorems can be used to both characterize their behavior and misbehavior.

4.1 Analytical Behavior Characterization

We now begin our development of an analytical theory of the behavior of SFFBACs. Each of our theorems will parallel a similar theorem from the single algorithm case, and we will see that, while the math and conditions become a great deal more tedious, the SFFBAC case is much like the single algorithm case for most of the

Table 4.1: Requirements on the accuracy of initialization to guarantee exponential stability of the SFFBAC. The critical $\beta = \beta_c$ is defined in (4.1).

α, β	$\mathbf{k_m} =$	$\mathbf{r_b} <$
$\alpha \neq \beta$	$\frac{\ln\left(\frac{\ln(\beta)}{\ln(\alpha)}\left(1-\frac{(\alpha-\beta)\|b_1-b_1^*\|}{\chi\|a_1-a_1^*\|}\right)\right)}{\ln(\frac{\alpha}{\beta})}$	$\beta^{k_m}\|b_1 - b^*\| + \frac{\chi(\alpha^{k_m}-\beta^{k_m})}{\alpha-\beta}$
$\alpha = \beta < \beta_c$	1	$\beta^{k_m}\|b_1 - b^*\| + k_m\beta^{k_m}\|a_1 - a^*\|$
$\alpha = \beta \geq \beta_c$	$\frac{-\|b_1-b^*\|}{\chi\|a_1-a^*\|} - \frac{1}{\ln(\beta)}$	$\beta^{k_m}\|b_1 - b^*\| + k_m\beta^{k_m}\|a_1 - a^*\|$

theorems. We will highlight the connections between the quantatative assumptions provided here and their qualitative counterparts provided by the divide and conquer conditions. We begin by developing a theorem similar to Theorem 1 for a non-disturbed and non-time-varying SFFBAC.

4.1.1 Deterministic Analysis Fixed Regime

Theorem 6 (SFFBACs designed with DaC). *Consider two adaptive elements connected in series feedforward form, as depicted in Figure 3.1*

$$a_{k+1} = f_a(a_k, x_k)$$

$$y_k = g_a(a_k, x_k)$$

$$b_{k+1} = f_b(b_k, y_k) = f_b(b_k, g_a(a_k, x_k))$$

Suppose that the first element has an adaptive state equation that is contractive towards a^* in its first argument with constant $\alpha < 1$ (Condition 3)

$$a^* = f(a^*, x_k) \quad \forall k$$

$$\mathcal{B}_{a^*} = \{\xi | \|\xi - a^*\| < r_a\}$$

$$\|f_a(\xi, x_k) - f_a(a^*, x_k)\| < \alpha\|\xi - a^*\| \quad \alpha < 1 \quad \forall k \quad \forall \xi \in \mathcal{B}_{a^*}$$

Also suppose that the second device has been designed in such a way that it is exponentially stable to b^*, by guaranteeing contractivity with constant $\beta < 1$ within a ball $\mathcal{B}_{b^*}$ as long as the first element is at its stationary point (Condition 4)

$$b^* = f_b(b^*, g(a^*, x_k)) \qquad \forall k$$

$$\mathcal{B}_{b^*} = \{\xi | \|\xi - b^*\| < r_b\}$$

$$\|f_b(\xi, g(a^*, x_k)) - f_b(b^*, g(a^*, x_k))\| < \beta\|\xi - b^*\| \quad \beta < 1 \quad \forall \xi \in \mathcal{B}_{b^*} \quad \forall k$$

Also suppose that the second element's update equation coupled with the first element's input output equation is Lipschitz continuous in a_k (Condition 5), such that

$$\|f_b(b, g_a(\xi_1, x_k)) - f_b(b, g_a(\xi_2, x_k))\| < \chi\|\xi_1 - \xi_2\| \quad \forall b \in \mathcal{B}_{b^*} \quad \forall \xi_1, \xi_2 \in \mathcal{B}_{a^*}$$

Suppose the initializations are accurate enough (Condition 6), such that they satisfy

$$\|a_1 - a^*\| < r_a$$

and a second condition which can be found as the entry in Table 4.1 corresponding to the values of α and β, i.e.

$$\beta_c = \exp\left\{\frac{-\chi\|a_1 - a^*\|}{\chi\|a_1 - a^*\| + \|b_1 - b^*\|}\right\} \tag{4.1}$$

If all of these assumptions hold, then the whole system is locally exponentially stable to a^* and b^*. For $\alpha \neq \beta$ the parameter errors are bounded by

$$\|a_{k+1} - a^*\| < \alpha^k\|a_1 - a^*\|$$

$$\|b_{k+1} - b^*\| < \beta^k\|b_1 - b^*\| + \frac{\chi(\alpha^k - \beta^k)}{\alpha - \beta}\|a_1 - a^*\|$$

and for $\alpha = \beta$ by

$$\|a_{k+1} - a^*\| < \alpha^k\|a_1 - a^*\| \tag{4.2}$$

$$\|b_{k+1} - b^*\| \leq \beta^k\|b_1 - b^*\| + \chi k\alpha^k\|a_1 - a^*\| = \beta^k\|b_1 - b^*\| + \chi k\beta^k\|a_1 - a^*\| \tag{4.3}$$

$\diamond$ We begin with the adaptive state equation

$$a_{k+1} - a^* = f_a(a_k, x_k) - a^*$$

Using the fact that a^* is a fixed point, we have

$$a_{k+1} - a^* = f_a(a_k, x_k) - f_a(a^*, x_k)$$

Taking the norm of both sides and using the Lipschitz continuity of the adaptive state equation, we have

$$\|a_{k+1} - a^*\| \leq \alpha \|a_k - a^*\|$$

Iterating this yields

$$\|a_{k+1} - a^*\| \leq \alpha^k \|a_1 - a^*\| \tag{4.4}$$

which is our bound over time for the deviation in the first block from its tuned signal, a^*.

We now begin analyzing the second block. Adding clever forms of zero to the adaptive state equation of the second adaptive element yields

$$b_{k+1} - b^* = f_b(b_k, g_a(a_k, x_k)) - f_b(b_k, g_a(a^*, x_k)) + f_b(b_k, g_a(a^*, x_k)) - b^*$$

Because b^* is a fixed point, we have

$$b_{k+1} - b^* = f_b(b_k, g_a(a_k, x_k)) - f_b(b_k, g_a(a^*, x_k)) + f_b(b_k, g_a(a^*, x_k)) - f_b(b^*, g_a(a^*, x_k))$$

Taking norms, using the triangle inequality, and using the Lipschitz continuity of the adaptive state equation, we have

$$\|b_{k+1} - b^*\| \leq \beta \|b_k - b^*\| + \chi \|a_k - a^*\|$$

Iterating this inequality yields

$$\|b_{k+1} - b^*\| \leq \beta^k \|b_1 - b^*\| + \chi \sum_{i=0}^{k-1} \beta^i \|a_{k-i} - a^*\|$$

Using (4.4) we have

$$\|b_{k+1} - b^*\| \leq \beta^k \|b_1 - b^*\| + \chi \sum_{i=0}^{k-1} \beta^i \alpha^{k-i-1} \|a_1 - a^*\|$$

$$\|b_{k+1} - b^*\| \leq \beta^k \|b_1 - b^*\| + \chi \alpha^{k-1} \sum_{i=0}^{k-1} \left(\frac{\beta}{\alpha}\right)^i \|a_1 - a^*\| \tag{4.5}$$

Using the sum of a finite geometric series[1], we have, for $\alpha \neq \beta$

$$\|b_{k+1} - b^*\| \leq \beta^k \|b_1 - b^*\| + \chi \alpha^{k-1} \frac{1 - \left(\frac{\beta}{\alpha}\right)^k}{1 - \frac{\beta}{\alpha}} \|a_1 - a^*\|$$

Doing some manipulation yields the bound provided by the theorem.

$$\|b_{k+1} - b^*\| \leq \beta^k \|b_1 - b^*\| + \chi \frac{\alpha^k - \beta^k}{\alpha - \beta} \|a_1 - a^*\|$$

For the special case where $\alpha = \beta$, (4.5) becomes

$$\|b_{k+1} - b^*\| \leq \beta^k \|b_1 - b^*\| + \chi k \alpha^k \|a_1 - a^*\| = \beta^k \|b_1 - b^*\| + \chi k \beta^k \|a_1 - a^*\| \tag{4.6}$$

To finish the proof, we must verify that our trajectories remained within the ball in which our solutions held. For the case $\alpha \neq \beta$, first rewrite the bound as

$$\|b_{k+1} - b^*\| \leq e^{\ln(\beta)k} \|b_1 - b_1^*\| + \frac{\chi(e^{\ln(\alpha)k} - e^{\ln(\beta)k})}{\alpha - \beta} \|a_1 - a_1^*\|$$

We now consider the right hand side as a function $u(x)$. Taking the derivative of this with respect to k gives

$$\frac{du(k)}{dk} = \ln(\beta) e^{\ln(\beta)k} \|b_1 - b_1^*\| + \frac{\chi \|a_1 - a_1^*\|}{\alpha - \beta} \left(\ln(\alpha) e^{\ln(\alpha)k} - \ln(\beta) e^{\ln(\beta)k}\right) \tag{4.7}$$

Regrouping terms on the right of (4.7) and setting it equal to zero in order to find the location of the extrema yields

$$\left(\|b_1 - b_1^*\| - \frac{\chi \|a_1 - a_1^*\|}{\alpha - \beta}\right) e^{\ln(\beta)k_{max}} + \frac{\ln(\alpha)}{\ln(\beta)} \frac{\chi \|a_1 - a_1^*\|}{\alpha - \beta} e^{\ln(\alpha)k_{max}} = 0$$

[1]See appendix C.

$$\left(\|b_1 - b_1^*\| - \frac{\chi\|a_1 - a_1^*\|}{\alpha - \beta}\right) e^{\ln(\beta)k_{max}} = -\frac{\ln(\alpha)}{\ln(\beta)} \frac{\chi\|a_1 - a_1^*\|}{\alpha - \beta} e^{\ln(\alpha)k_{max}} \tag{4.8}$$

$$\frac{\ln(\beta)}{\ln(\alpha)} \left(1 - \frac{(\alpha - \beta)\|b_1 - b_1^*\|}{\chi\|a_1 - a_1^*\|}\right) = e^{\ln(\frac{\alpha}{\beta})k_{max}}$$

So the time at which the system reaches is maximum error is

$$k_{max} = \frac{\ln\left(\frac{\ln(\beta)}{\ln(\alpha)}\left(1 - \frac{(\alpha-\beta)\|b_1 - b_1^*\|}{\chi\|a_1 - a_1^*\|}\right)\right)}{\ln(\frac{\alpha}{\beta})}$$

To verify that this is indeed a maximum, we need to look at the sign of the second derivative. Taking the derivative of (4.8), we get

$$\frac{d^2u(k)}{dk^2} = \ln(\beta)^2 \left(\|b_1 - b_1^*\| - \frac{\chi\|a_1 - a_1^*\|}{\alpha - \beta}\right) e^{\ln(\beta)k} + \ln(\alpha)^2 \frac{\chi\|a_1 - a_1^*\|}{\alpha - \beta} e^{\ln(\alpha)k}$$

We then evaluate this at $k = k_{max}$ and substitute in (4.8) for the term in parenthesis on the left.

$$\ln(\beta)^2 \left(-\frac{\ln(\alpha)}{\ln(\beta)} \frac{\chi\|a_1 - a_1^*\|}{\alpha - \beta} e^{\ln(\alpha)k_{max}}\right) + \ln(\alpha)^2 \frac{\chi\|a_1 - a_1^*\|}{\alpha - \beta} e^{\ln(\alpha)k_{max}}$$

Rearranging terms yields a form of the second derivative more convenient for sign analysis

$$\underbrace{\ln(\alpha)}_{<0} \underbrace{\frac{\ln(\alpha) - \ln(\beta)}{\alpha - \beta}}_{>0} \underbrace{\left(\chi\|a_1 - a_1^*\| e^{\ln(\alpha)k_{max}}\right)}_{>0} < 0$$

This verifies that the second derivative is negative, and thus that we have a maximum at $k = k_{max}$. We need this maximum error small enough that we can guarantee that the two adaptive elements remain within the balls in which our assumptions are valid. Thus, we require

$$\beta^{k_{max}}\|b_1 - b_1^*\| + \frac{\chi(\alpha^{k_{max}} - \beta^{k_{max}})}{\alpha - \beta}\|a_1 - a_1^*\| < r_b$$

and

$$\|a_1 - a_1^*\| < r_a$$

whenever $\alpha \neq \beta$. These are the two conditions required by the theorem as shown in Table 4.1.

The final case we must check is the special case that $\alpha = \beta$. Defining the right side of (4.6) to be a function $w(k)$ and taking its derivative and setting it equal to zero gives

$$\frac{dw(k)}{dk}|_{k=k_m} = \ln(\beta)\beta^{k_m}\|b_1 - b^*\| + \chi\ln(\beta)k_m\beta^{k_m}\|a_1 - a^*\| + \chi\beta^{k_m}\|a_1 - a^*\| = 0$$

$$-\ln(\beta)\beta^{k_m}\|b_1 - b^*\| - \chi\beta^{k_m}\|a_1 - a^*\| = \chi\ln(\beta)k_m\beta^{k_m}\|a_1 - a^*\|$$

$$k_m = \frac{-\ln(\beta)\|b_1 - b^*\| - \chi\|a_1 - a^*\|}{\chi\ln(\beta)\|a_1 - a^*\|}$$

$$k_m = \frac{-\|b_1 - b^*\|}{\chi\|a_1 - a^*\|} - \frac{1}{\ln(\beta)} \tag{4.9}$$

Because we start at $k = 1$, we see that we will only encounter this extremum if

$$\frac{-\|b_1 - b^*\|}{\chi\|a_1 - a^*\|} - \frac{1}{\ln(\beta)} \geq 1$$

$$-\frac{1}{\ln(\beta)} \geq 1 + \frac{\|b_1 - b^*\|}{\chi\|a_1 - a^*\|}$$

$$\ln(\beta) \geq \frac{-\chi\|a_1 - a^*\|}{\chi\|a_1 - a^*\| + \|b_1 - b^*\|}$$

$$\beta \geq \exp\left\{\frac{-\chi\|a_1 - a^*\|}{\chi\|a_1 - a^*\| + \|b_1 - b^*\|}\right\}$$

Thus, if $\alpha = \beta < \exp\left\{\frac{-\chi\|a_1 - a^*\|}{\chi\|a_1 - a^*\| + \|b_1 - b^*\|}\right\}$ all we require is that the initial conditions put us into the correct region to begin with

$$\|a_1 - a^*\| < r_a$$

$$\|b_1 - b^*\| + \chi\|a_1 - a^*\| < r_b$$

Continuing on with the case when $\alpha = \beta \geq \exp\left\{\frac{-\chi\|a_1 - a^*\|}{\chi\|a_1 - a^*\| + \|b_1 - b^*\|}\right\}$, we must verify that k_m is indeed a local maximum. To do so, we take the second derivative of

$w(k)$ and check its sign.

$$\begin{aligned}\frac{d^2w(k)}{dk^2}|_{k=k_m} = \;& \ln(\beta)^2\beta^{k_m}\|b_1 - b^*\| + \chi\ln(\beta)^2 k_m\beta^{k_m}\|a_1 - a^*\| \\ & +\chi\ln(\beta)\beta^{k_m}\|a_1 - a^*\| + \chi\ln(\beta)\beta^{k_m}\|a_1 - a^*\| = 0\end{aligned}$$

Substituting in the expression for k_m from (4.9) gives the equation

$$\begin{aligned}\frac{d^2w(k)}{dk^2}|_{k=k_m} &= \ln(\beta)^2\|b_1 - b^*\| + 2\ln(\beta)\chi\|a_1 - a^*\| - \ln(\beta)^2\|b_1 - b^*\| \\ & - \ln(\beta)\chi\|a_1 - a^*\| \\ &= \ln(\beta)\chi\|a_1 - a^*\| < 0\end{aligned}$$

where we have ignored the trivial case when the first adaptive element is started exactly at its equilibrium. Since the second derivative is negative here, k_m is a maximum. Thus, if $\alpha = \beta \geq \exp\left\{\frac{-\chi\|a_1 - a^*\|}{\chi\|a_1 - a^*\| + \|b_1 - b^*\|}\right\}$, requiring

$$\|a_1 - a^*\| < r_a$$

$$\beta^{k_m}\|b_1 - b^*\| + \chi k_m\beta^{k_m}\|a_1 - a^*\| < r_b$$

guarantees that we remain within the ball in which our assumptions were valid.◇

This theorem provides us with our basic result that our other results will build on. Namely, if the first adaptive element in the SFFBACs is contractive to its desired point, and if the second adaptive element is contractive to its desired point when the first element is fixed at its desired point, then both elements are locally exponentially stable to their desired points. The following example illustrates this theorem.

Example 4.1 (A Series Feedforward Binary Adaptive Compound). *Consider two adaptive elements connected together in series feedfoward form, as in Figure 3.1, with adaptive state equations*

$$a_{k+1} = a_k - \mu_a\,(a_k - 1)$$

$$b_{k+1} = b_k - \mu_b \left(b_k - 3a_k + 1\right)$$

We wish to verify that the conditions of Theorem 6 hold true for this system. First of all, we note that $a^* = 1$ *and* $b^* = 2$. *Then, calculating the contractivity and Lipschitz constants*

$$\alpha = \max_{a_1 \in \mathcal{B}_{a^*}} \frac{\|f_a(a_1, x_k) - a^*\|}{\|a_1 - a^*\|} = \max_{a_1 \in B_{a^*}} (1 - \mu_a) = (1 - \mu_a)$$

$$\beta = \max_{b_1 \in \mathcal{B}_{b^*}} \frac{\|f_b(b_1, g_a(a^*, x_k)) - b^*\|}{\|b_1 - b^*\|} = (1 - \mu_b)$$

$$\chi = \max_{a_1 \in \mathcal{B}_{a^*}} \frac{\|f_b(b, g(a_1, x_k)) - f_b(b, g(a^*, x_k))\|}{\|a_1 - a^*\|} = 3\mu_b$$

We see that this system (due to its linearity) can be contractive within any ball we choose and satisfies the conditions of the theorem. Figure 4.1 compares the bound offered by the theorem and the actual evolution of the trajectory. We see that, for this simple linear system, the bound offered by the theorem is very tight, because the solid and dotted curves are indistinguishable.

4.1.2 Deterministic Analysis Time-Varying Regime

Next we extend our analysis to handle time variation and disturbances, but still assume that we have a contraction at every step. Once again, we highlight the connections with the qualitative DAC conditions. The theorem we will develop is similar to Theorem 3.

Theorem 7 (Time Variation and Disturbances in DaC SFFBACs). *Consider two adaptive elements connected in series feed-forward form, as in Figure 3.1, and suffering from disturbances (noise) and time variation*

$$a_{k+1} = f_a(k, a_k, x_k) + n_k^a$$

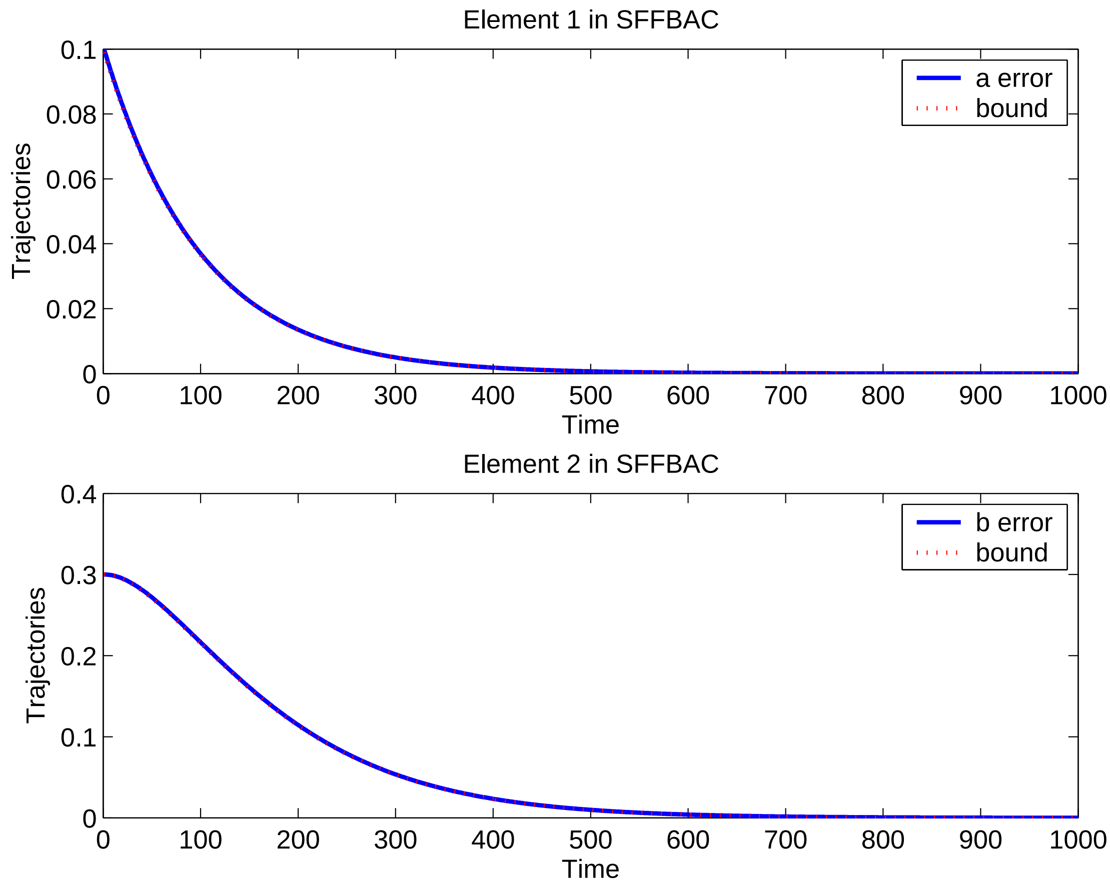

Figure 4.1: Example of exponential stability of a SFFBAC. Note that the bounds are very tight in this example, because the updates are linear.

Table 4.2: Requirements on the accuracy of initialization to guarantee exponential stability to within a ball of the desired points for the SFFBAC subject to disturbances and time variation. The critical β is $\beta_c = \exp\left\{\frac{-\chi\|a_1-a^*\|}{\chi\|a_1-a^*\|+\|b_1-b^*\|}\right\}$.

α, β	$\mathbf{k_m} =$	$\mathbf{r_b} <$
$\alpha \neq \beta$	$\frac{\ln\left(\frac{\ln(\beta)}{\ln(\alpha)}\left(1-\frac{(\alpha-\beta)\|b_1-b_1^*\|}{\chi\|a_1-a_1^*\|}\right)\right)}{\ln(\frac{\alpha}{\beta})}$	$\beta^{k_m}\|b_1-b^*\| + \frac{\chi(\alpha^{k_m}-\beta^{k_m})}{\alpha-\beta} + \frac{c_a+d_1}{(1-\beta)(1-\alpha)} + \frac{c_b+d_2}{(1-\beta)}$
$\alpha = \beta < \beta_c$	1	$\beta^{k_m}\|b_1-b^*\| + k_m\beta^{k_m}\|a_1-a^*\| + \frac{c_a+d_1}{(1-\beta)(1-\alpha)} + \frac{c_b+d_2}{(1-\beta)}$
$\alpha = \beta \geq \beta_c$	$\frac{-\|b_1-b^*\|}{\chi\|a_1-a^*\|} - \frac{1}{\ln(\beta)}$	$\beta^{k_m}\|b_1-b^*\| + k_m\beta^{k_m}\|a_1-a^*\| + \frac{c_a+d_1}{(1-\beta)(1-\alpha)} + \frac{c_b+d_2}{(1-\beta)}$

$$y_k = g_a(a_k, x_k)$$

$$b_k = f_b(k, b_k, y_k) + n_k^b = f_b(k, b_k, g_a(a_k, x_k)) + n_k^b$$

with DaC-designed stationary point trajectories a_k^* and b_k^*

$$a_k^* = f_a(k, a_k^*, x_k) \quad \forall k$$

$$b_k^* = f_b(k, b_k^*, g_a(a_k^*, x_k)) \quad \forall k$$

and with the DaC contractivity (3 and 4) conditions.

$$\|f_a(\xi, x_k) - f_a(a_k^*, x_k)\| < \alpha\|\xi - a_k^*\| \quad \alpha < 1 \quad \forall\xi \in \mathcal{B}_{a_k^*} \quad \forall k \tag{4.10}$$

$$\|f_b(\xi, g_a(a_k^*, x_k)) - f_b(b_k^*, g_a(a_k^*, x_k))\| < \beta\|\xi - b_k^*\| \quad \beta < 1 \quad \forall\xi \in \mathcal{B}_{b_k^*} \quad \forall k \tag{4.11}$$

Where we have defined $\mathcal{B}_{a_k^*} = \{\xi_k \| \|\xi_k - a_k^*\| < r_a\}$ and $\mathcal{B}_{b_k^*} = \{\xi_k \| \|\xi_k - b_k^*\| < r_b\}$. Also assume lipschitz continuity in the coupling of first element's input output equation and the second elements adaptive state equation (Condition 5)

$$\|f_b(b, g_a(\xi_1, x_k)) - f_b(b, g_a(\xi_2, x_k))\| < \chi\|\xi_1 - \xi_2\| \quad \forall\xi_1, \xi_2 \in \mathcal{B}_{a_k^*} \quad \forall b \in \mathcal{B}_{a_k^*} \tag{4.12}$$

and small enough noise and slow enough time variation (Condition 6).

$$\|a^*_{k+1} - a^*_k\| < d_1 \quad \|b^*_{k+1} - b^*_k\| < d_2$$

$$\|n^a_k\| < c_a \quad \|n^b_k\| < c_b$$

Then, if the initializations are accurate enough, the disturbance is small enough, and the time variation is slow enough (Condition 6), such that

$$\frac{d_1 + c_a}{1-\alpha} + \|a_1 - a^*_1\| < r_a$$

and the initialization error satisfies the bound corresponding to α and β in Table 4.2, then the whole system is exponentially stable to a ball of size $\frac{d_1+c_a}{1-\alpha}$ around a^*_k and a ball of size $\frac{d_2+c_b}{1-\beta} + \frac{\chi(d_1+c_a)}{(1-\alpha)(1-\beta)}$ around b^*_k. For the case that $\beta \neq \alpha$, the error is bounded by

$$\|a_{k+1} - a^*_{k+1}\| \leq \alpha^k \|a_1 - a^*_1\| + \frac{d_1 + c_a}{1-\alpha}$$

$$\|b_{k+1} - b^*_{k+1}\| \leq \beta^k \|b_1 - b^*_1\| + \frac{\chi(\alpha^k - \beta^k)}{\alpha - \beta}\|a_1 - a^*_1\| + \frac{d_2 + c_b}{1-\beta} + \frac{\chi(d_1 + c_a)}{(1-\alpha)(1-\beta)}$$

For the case that $\beta = \alpha$, the error is bounded by

$$\|a_{k+1} - a^*_{k+1}\| \leq \alpha^k \|a_1 - a^*_1\| + \frac{d_1 + c_a}{1-\alpha}$$

$$\|b_{k+1} - b^*_{k+1}\| \leq \beta^k \|b_1 - b^*_1\| + \chi k \beta^k \|a_1 - a^*_1\| + \frac{d_2 + c_b}{1-\beta} + \frac{\chi(d_1 + c_a)}{(1-\alpha)(1-\beta)}$$

$\diamond$ As usual, we begin with an adaptive state equation to which we have added a clever form of zero.

$$a_{k+1} - a^*_{k+1} = f_a(k, a_k, x_k) - a^*_k + a^*_k - a^*_{k+1} + n^a_k$$

Taking norms of both sides and using the triangle inequality yields

$$\|a_{k+1} - a^*_{k+1}\| \leq \|f_a(k, a_k, x_k) - f_a(k, a^*_k, x_k)\| + \|a^*_k - a^*_{k+1}\| + \|n^a_k\|$$

Using the assumption we made about the contractivity of the first element's adaptive state equation, we have

$$\|a_{k+1} - a^*_{k+1}\| \leq \alpha\|a_k - a^*_k\| + \|a^*_k - a^*_{k+1}\| + \|n^a_k\|$$

Running the recursion and replacing the disturbances and time variation with their bounds we have

$$\|a_{k+1} - a^*_{k+1}\| \leq \alpha^k\|a_1 - a^*_1\| + \sum_{i=0}^{k} \alpha^i(d_1 + c_a)$$

Since the sum above is a finite geometric series, and α is a positive parameter, it will be less than the corresponding infinite geometric series, which gives us our next equation

$$\|a_{k+1} - a^*_{k+1}\| \leq \alpha^k\|a_1 - a^*_1\| + \frac{d_1 + c_a}{1 - \alpha}$$

which is the bound that we predicted on the error in the first adaptive element from its tuned trajectory. Continuing on to the second adaptive element we have

$$\begin{aligned} b_{k+1} - b^*_{k+1} = \; & f_b(k, b_k, g_a(a_k, x_k)) - f_b(k, b_k, g_a(a_k, x_k)) + f_b(k, b_k, g_a(a_k, x_k)) \\ & -b^*_k + b^*_k - b^*_{k+1} + n^b_k \end{aligned}$$

Taking norms and using the triangle inequality gives

$$\begin{aligned} \|b_{k+1} - b^*_{k+1}\| \leq \; & \|f_b(k, b_k, g_a(a_k, x_k)) - f_b(k, b_k, g_a(a^*_k, x_k))\| + \\ & \|f_b(k, b_k, g_a(a^*_k, x_k)) - f_b(k, b^*_k, g_a(a^*_k, x_k))\| + \|b^*_k - b^*_{k+1}\| + \|n^b_k\| \end{aligned}$$

Using the assumed Lipschitz continuities we have

$$\|b_{k+1} - b^*_{k+1}\| \leq \chi\|a_k - a^*_k\| + \beta\|b_k - b^*_k\| + \|b^*_k - b^*_{k+1}\| + \|n^b_k\|$$

Running the recursion gives

$$\|b_{k+1} - b^*_{k+1}\| \leq \beta^k\|b_1 - b^*_1\| + \sum_{i=0}^{k-1} \beta^i \left(\chi\|a_{k-i} - a^*_{k-i}\| + \|b^*_{k-i} - b^*_{k-i+1}\| + \|n^b_{k-i}\|\right)$$

Substituting in (4.1.2), we have

$$\|b_{k+1} - b^*_{k+1}\| \leq \beta^k \|b_1 - b^*_1\| + \sum_{i=0}^{k-1} \beta^i \chi \left(\alpha^{k-i-1} \|a_1 - a^*_1\| + \frac{d_1 + c_a}{1-\alpha} \right) + \frac{d_2 + c_b}{1-\beta}$$

Using once again the sum of a finite geometric series and the sum of an infinite geometric series as we did in the previous theorem, for the case that $\alpha \neq \beta$ we have

$$\|b_{k+1} - b^*_{k+1}\| \leq \beta^k \|b_1 - b^*_1\| + \frac{\chi(\alpha^k - \beta^k)}{\alpha - \beta} \|a_1 - a^*_1\| + \frac{d_1 + c_a}{(1-\alpha)(1-\beta)} + \frac{d_2 + c_b}{1-\beta}$$

and for the case when $\alpha = \beta$, we have

$$\|b_{k+1} - b^*_{k+1}\| \leq \beta^k \|b_1 - b^*_1\| + \chi k \beta^k \|a_1 - a^*_1\| + \frac{d_1 + c_a}{(1-\alpha)(1-\beta)} + \frac{d_2 + c_b}{1-\beta} \quad (4.13)$$

Which is our final desired bound on the parameter error in the second adaptive element. We note that the misadjustment in the first block causes misadjustment in the second block through the sensitivity, χ. Thus, in the second adaptive element, the size of the ball within which we can guarantee that our solution lies is affected by the size of the ball in which the first adaptive element's parameters are known to lie.

To finish the proof, we need to verify that we never left the ball in which our assumptions valid. To do so it is necessary to determine our maximum error. For $\alpha \neq \beta$

$$\begin{aligned} \max_k \|b_{k+1} - b^*_{k+1}\| \leq \; & \max_k \Big(\beta^k \|b_1 - b^*_1\| + \tfrac{\chi(\alpha^k - \beta^k)}{\alpha - \beta} \|a_1 - a^*_1\| + \tfrac{d_1 + c_a}{(1-\alpha)(1-\beta)} \\ & + \tfrac{d_2 + c_b}{1-\beta} \Big) \end{aligned}$$

Taking the derivative with respect to k allows us to perform the maximization. First, rewrite the bound as

$$e^{\ln(\beta)k} \|b_1 - b^*_1\| + \frac{\chi(e^{\ln(\alpha)k} - e^{\ln(\beta)k})}{\alpha - \beta} \|a_1 - a^*_1\| + \frac{d_1 + c_a}{(1-\alpha)(1-\beta)} + \frac{d_2 + c_b}{1-\beta}$$

Taking the derivative with respect to k gives

$$\ln(\beta)e^{\ln(\beta)k}\|b_1 - b_1^*\| + \frac{\alpha\chi\|a_1 - a_1^*\|}{\alpha - \beta}\left(\ln(\alpha)e^{\ln(\alpha)k} - \ln(\beta)e^{\ln(\beta)k}\right)$$

which is the same as (4.7). We have already characterized the location of the zeros of this function when we studied the zeros of (4.7). Furthermore, since the second derivatives will be the same as well, we know that k_{max} from the previous theorem will be a local maximum for the bound on the error in this theorem as well. Thus, we can use our results for k_{max} from the previous theorem for the case that $\alpha \neq \beta$. We must also consider the special case when $\alpha = \beta$. In this instance, we can apply the same derivation as in the previous theorem to get k_m, since the derivative of (4.13) with respect to k is exactly the same as the derivative of (4.6) which was for the special case $\alpha = \beta$ in the previous theorem. Thus, using the k_m from the previous theorem and our bound on the parameter error, we can produce the initialization error bounds required in Table 4.2. $\diamond$

Qualitatively, this theorem assures us that our SFFBAC will be robust to small disturbances and perturbations when we initialize it accurately enough. Essentially, all we need to guarantee this is contractivity of the undisturbed and time-invariant system, and continuity in the second adaptive state equation with respect to the first elements adaptive state. We provide an example here to highlight the quantitative aspects of the theorem. Not surprisingly, the theorem shows that (possible[2]) jitter in the first element couples through to (possible) jitter in the second element's adaptive state. This fits with intuition: since the first element processes the input to the second element, if it does so in a noisy way, it is possible that the second element will be sensitive to this noise. The parameter χ is a

[2]The theorem provides an upper bound on the error, which is not necessarily the achieved supremum.

measure of this sensitivity: second elements which are more sensitive to changes in the first elements parameters have larger χs and possibly have more jitter due to fluctuations in the first element's state. Of course, we have not characterized the disturbances yet, and, thus we have not exploited fully the possibility that they might be zero on average. Our next two theorems specialize our analysis to this case. But first, we consider a numerical example to highlight the quantitative aspects of the previous theorem.

Example 4.2 (SFFBAC Time Varying and Disturbances). *Consider two adaptive elements connected together in series feedforward form, as in Figure 3.1, with adaptive state equations*

$$a_{k+1} = a_k - \mu_a \left(a_k - a_k^*\right) + n_k$$

$$b_{k+1} = b_k - \mu_b \left(b_k - 3a_k - b_k^*\right) + n_{k,2}$$

We wish to verify the accuracy of Theorem 7. To begin, we calculate the Lipschitz and contractivity constants, which are $\alpha = (1-\mu_a)$, $\beta = (1-\mu_b)$, as well as $\chi = 3\mu_b$ since the algorithm is in a simple linear form. Furthermore, the linear form allows these constants to be globally true. As we can see in Figure 4.2, the bound offered by this Theorem is not tight at all. Thus, we will develop averaging theory in the next two sections to allow us to more fully study the way the two adaptive elements interact, as well as get a better handle on how the step sizes affect convergent jitter in an averaging form algorithm. Our current bounds do not indicate that turning our step size down will decrease the bound due to the disturbance, while our intuition and practical experience with averaging form algorithms suggest that such a situation is indeed the case. Deterministic averaging theory will provide us with a method of confirming our intuition about this concept, and of showing that

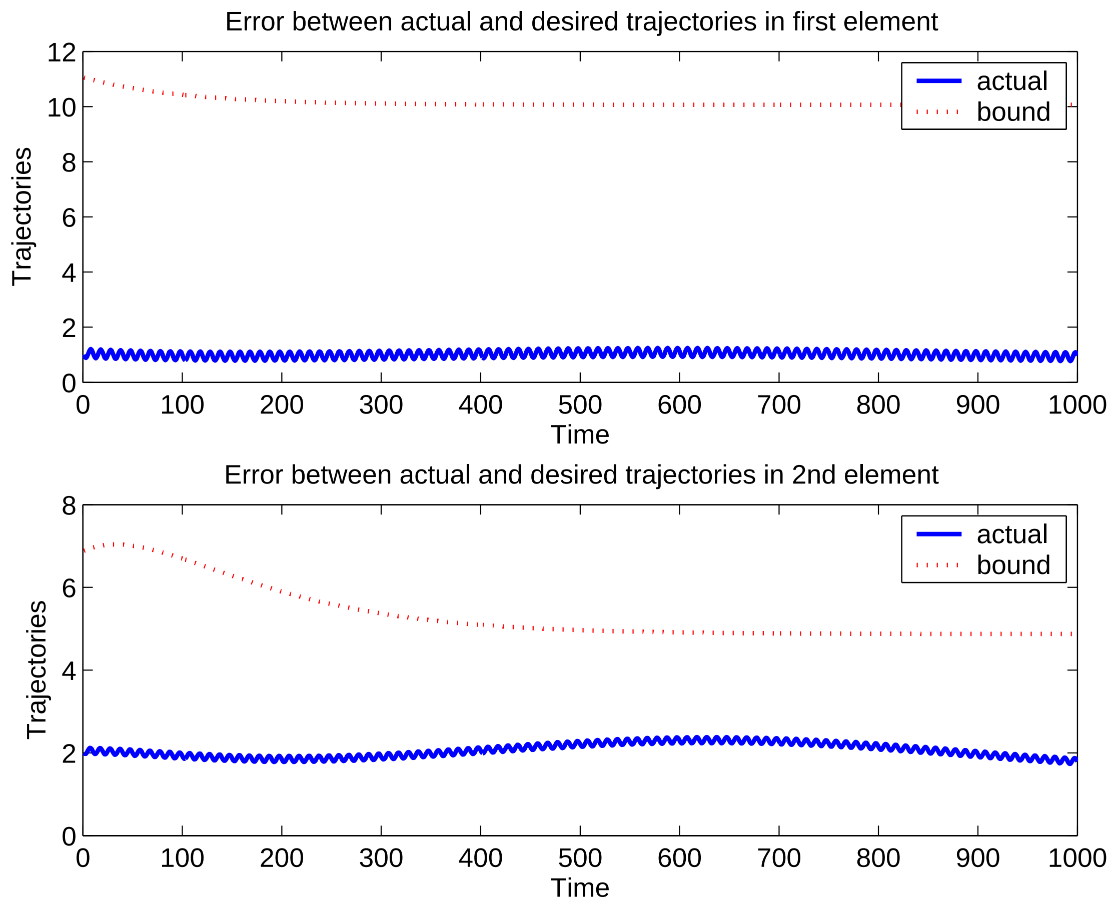

Figure 4.2: Series Feed-forward Binary Adaptive Compound with disturbances and time variation.

it does indeed apply to the series feed-forward binary adaptive compound case as well.

4.1.3 SFFBAC Deterministic Finite Time Averaging Theorem

Now we extend our analysis to include the possibility that we might not have a contraction towards the desired points at every step, even when the first element is at its desired state, while on average there will still be a contraction. The theorem we will develop will be similar to Theorem 4.

Theorem 8 (SFFBAC Finite Time Averaging). *Consider two adaptive elements connected as in Figure 3.1 to form a series feed-forward binary adaptive compound with the averaging form adaptive state equations*

$$\hat{a}_{k+1} = \hat{a}_k + \mu_a \hat{f}_a(k, \hat{a}_k, x_k) \tag{4.14}$$

$$\hat{b}_{k+1} = \hat{b}_k + \mu_b \hat{f}_b(k, \hat{b}_k, g_a(\hat{a}_k, x_k)) \tag{4.15}$$

and its associated error form

$$\begin{aligned} \tilde{a}_{k+1} &= \tilde{a}_k + \mu_a f_a(k, \tilde{a}_k, x_k) + a_k^* - a_{k+1}^* \\ \tilde{b}_{k+1} &= \tilde{b}_k + \mu_b f_b(k, \tilde{b}_k, \tilde{a}_k) + b_k^* - b_{k+1}^* \end{aligned} \tag{4.16}$$

where $\tilde{a}_k = \hat{a}_k - a_k^*$, $f_a(k, \tilde{a}_k) = \hat{f}_a(k, \tilde{a}_k + a_k^*)$, $\tilde{b}_k = \hat{b}_k - b_k^*$, *and* $f_b(k, \tilde{b}_k, \tilde{a}_k) = \hat{f}_b(k, \tilde{b}_k + b_k^*, g_a(\tilde{a}_k + a_k^*, x_k))$. *Suppose that the following averaged update functions exist*

$$f_{av,a}(\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f_a(i, \xi, x_i)$$
$$f_{av,b}(\nu, \xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f_b(i, \nu, \xi)$$

Which give the following averaged systems

$$\begin{aligned} \bar{a}_{k+1} &= \bar{a}_k + \mu_a f_{av,a}(\bar{a}_k) + a_k^* - a_{k+1}^* \\ \bar{b}_{k+1} &= \bar{b}_k + \mu_b f_{av,b}(\bar{b}_k, \bar{a}_k) + b_k^* - b_{k+1}^* \end{aligned} \tag{4.17}$$

Suppose that the averaged systems, (4.14) and (4.15), obey the conditions in Theorem 7, so that they are exponentially stable to balls around the trajectories a_k^* *and* b_k^*. *Specifically, the averaged error system must satisfy conditions related to (4.10) and (4.11) (DAC Conditions 3 and 4)*

$$\|\bar{a}_k + \mu_a f_{av}(\bar{a}_k)\| \leq \alpha \|\bar{a}_k\|$$

$$\|\bar{b}_k + \mu_b f_{av,b}(\bar{b}_k, 0)\| \leq \beta \|\bar{b}_k\|$$

and related to (4.12) (DAC Condition 5)

$$\|\nu + \mu_b f_{av,b}(\nu, \xi_1) - (\nu + \mu_b f_{av,b}(\nu, \xi_2))\,\|$$
$$= |\mu_b| \|f_{av,b}(\nu, \xi_1) - f_{av,b}(\nu, \xi_2)\| \leq \mu_b \gamma \|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2 \in \mathcal{B}_0(h_a)\ \forall \nu \in \mathcal{B}_0(h_b)$$

Also suppose the unaveraged $\tilde{a}_k$ system, (4.14), obeys the conditions of Theorem 5, so that the averaging error can be bounded linearly in μ_a. Define the total perturbation for the second element to be

$$p(k, \nu, \xi) = \sum_{i=1}^{k} [f_b(i, \nu, \xi) - f_{av,b}(\nu, \xi)]$$

and assume it is Lipschitz continuous uniformly in both arguments and bounded at the desired states such that

$$\|p(k, \nu_1, \xi) - p(k, \nu_2, \xi)\| < L_{p,b}\|\nu_1 - \nu_2\| \quad \forall \nu_1, \nu_2 \in \mathcal{B}_0(h_b), \quad \xi \in \mathcal{B}_0(h_a) \quad (4.18)$$

$$\|p(k, \nu, \xi_1) - p(k, \nu, \xi_2)\| < \chi \|\xi_1 - \xi_2\| \quad \forall \nu \in \mathcal{B}_0(h_b), \quad \xi_1, \xi_2 \in \mathcal{B}_0(h_a) \quad (4.19)$$

$$\|p(k, 0, 0)\| \leq B_{p,b} \quad \forall k \in \{1, 2, \cdots, T_b/\mu_b\} \quad (4.20)$$

where $\mathcal{B}_0(h_b) = \{\nu | \|\nu\| < h_b\}$, $\mathcal{B}_0(h_a) = \{\xi | \|\xi\| < h_a\}$, and $h_a = h$, where h is the size of the balls used in Theorem 4. Assume that $f_{av,b}(b, a)$ is locally Lipschitz such that

$$\|f_{av,b}(\xi_1, \nu) - f_{av,b}(\xi_2, \nu)\| < L_{f,b}\|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2 \in \mathcal{B}_0(h_b) \quad \forall \nu \in \mathcal{B}_0(h_a)$$

$$\|f_{av,b}(\nu, \xi_1) - f_{av,b}(\nu, \xi_2)\| \leq \gamma \|\xi_1 - \xi_2\| \quad \forall \nu \in \mathcal{B}_0(h_b) \quad \forall \xi_1, \xi_2 \in \mathcal{B}_0(h_a)$$

where $\mathcal{B}_0(h_a) = \{\xi | \|\xi\| < h_a\}$ and $\mathcal{B}_0(h_b) = \{\nu | \|\nu\| < h_b\}$. Furthermore, assume that the second element's unaveraged adaptive state equation is bounded, such that

$$\|f_b(k, \nu_k, g_a(\xi_k, x_k))\| < B_{f,b} \quad \forall k \quad \forall \nu_k \in \mathcal{B}_0(h_b) \quad \forall \xi_k \in \mathcal{B}_0(h_a)$$

Also assume that the step sizes are small enough and the time variation is slow enough that

$$\|a^*_{k+1} - a^*_k\| \le c_a \qquad \|b^*_{k+1} - b^*_k\| \le c_b$$

and

$$\begin{aligned} h_b > \quad & \beta^{k_m(\alpha,\beta)}\|b_1\| + d(k_m(\alpha,\beta),\alpha,\beta)\|a_1\| + \tfrac{c_a}{(1-\alpha)(1-\beta)} + \tfrac{c_b}{1-\beta} \\ & + e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. + \mu_a T_b \chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b}c_b\right] \end{aligned}$$

where

$$d(k,\alpha,\beta) = \begin{cases} \frac{\mu_b\gamma(\alpha^k-\beta^k)}{\alpha-\beta} & \alpha \ne \beta \\ \mu_b\gamma k\beta^k & \alpha = \beta \end{cases}$$

and

$$k_m(\alpha,\beta) = \begin{cases} \frac{\ln\left(\frac{\ln(\beta)}{\ln(\alpha)}\left(1-\frac{(\alpha-\beta)\|\tilde{b}_1\|}{\chi\|\tilde{a}_1\|}\right)\right)}{\ln(\frac{\alpha}{\beta})} & \alpha \ne \beta \\ \frac{-\|\tilde{b}_1\|}{\chi\|\tilde{a}_1\|} - \frac{1}{\ln(\beta)} & \alpha = \beta \ge \exp\left\{\frac{-\chi\|\tilde{a}_1\|}{\chi\|\tilde{a}_1\|+\|\tilde{b}_1\|}\right\} \\ 1 & \alpha = \beta < \exp\left\{\frac{-\chi\|\tilde{a}_1\|}{\chi\|\tilde{a}_1\|+\|\tilde{b}_1\|}\right\} \end{cases}$$

and $w(\mu_a, T)$ is the bound on the difference between the averaged and unaverged systems' trajectories in the second adaptive element ($\|\tilde{a}_k - \bar{a}_k\|$) from Theorem 5

$$w(\mu_a,T) = \frac{2-\alpha^{T/\mu}}{1-\alpha^{T/\mu}}\left(cT + \mu(B_p + L_p h + L_p B_f T)\right)e^{\lambda_f T}$$

If all of these assumptions hold, then over the time interval $\{1,...,T_b/\mu_b\}$, the difference between the averaged trajectory and the unaveraged trajectory, $\Delta_k = \|\tilde{b}_k - \bar{b}_k\|$, is bounded by a function which can be made arbitrarily small by shrinking BOTH μ_a and μ_b

$$\begin{aligned} \Delta_{k+1} \le \quad & e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. + \mu_a T_b\chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b}c_b\right] \\ & \forall k \in \{1,...,T_b/\mu_b\} \end{aligned} \tag{4.21}$$

$\diamond$ Subtracting the averaged system, (4.15), from the unaveraged system, (4.17), yields

$$\begin{aligned}\tilde{b}_{k+1}-\bar{b}_{k+1}= \;& \tilde{b}_1-\bar{b}_1+\mu_b\textstyle\sum_{i=1}^{k}\left[f_b(i,\tilde{b}_i,g_a(\tilde{a}_i,x_i))-f_{av,b}(\tilde{b}_i,\tilde{a}_i)\right]\\ &+\textstyle\sum_{i=1}^{k}\left[f_{av,b}(\tilde{b}_i,\tilde{a}_i)-f_{av,b}(\bar{b}_i,\bar{a}_i)\right]\end{aligned}$$

We observe again that

$$p(k,\tilde{a}_k,\tilde{b}_k)-p(k-1,\tilde{a}_k,\tilde{b}_k)=f_b(k,\tilde{b}_k,g_a(\tilde{a}_k,x_k))-f_{av,b}(\tilde{a}_k,\tilde{b}_k)\quad k\geq 2$$

which allows us to rewrite (4.1.3) as

$$\begin{aligned}\tilde{b}_{k+1}-\bar{b}_{k+1}= \;& \tilde{b}_1-\bar{b}_1+\mu_b\textstyle\sum_{i=2}^{k}\left[p(i,\tilde{b}_i,\tilde{a}_i)-p(i-1,\tilde{b}_i,\tilde{a}_i)\right]\\ &+\textstyle\sum_{i=1}^{k}\left[f_{av,b}(a_i,b_i)-f_{av,b}(\bar{a}_i,\bar{b}_i)\right]+f_b(\tilde{b}_1,\tilde{a}_1)-f_{av,b}(\tilde{b}_1,\tilde{a}_1)\end{aligned}$$

A change of indices of summation yields

$$\begin{aligned}\tilde{b}_{k+1}-\bar{b}_{k+1}= \;& \tilde{b}_1-\bar{b}_1+\mu_b\textstyle\sum_{i=1}^{k-1}\left[p(i,\tilde{b}_i,\tilde{a}_i)-p(i,\tilde{b}_{i+1},\tilde{a}_{i+1})\right]\\ &+\mu_b\textstyle\sum_{i=1}^{k}\left[f_{av,b}(\tilde{a}_i,\tilde{b}_i)-f_{av,b}(\bar{a}_i,\bar{b}_i)\right]+\mu_b p(k,\tilde{b}_k,\tilde{a}_k)\end{aligned}$$

Adding a clever form of zero yields

$$\begin{aligned}\tilde{b}_{k+1}-\bar{b}_{k+1}= \;& \tilde{b}_1-\bar{b}_1+\mu_b\textstyle\sum_{i=1}^{k-1}\Big[p(i,\tilde{b}_i,\tilde{a}_i)-p(i,\tilde{b}_{i+1},\tilde{a}_i)+p(i,\tilde{b}_{i+1},\tilde{a}_i)\\ &-p(i,\tilde{b}_{i+1},\tilde{a}_{i+1})\Big]+\mu_b\textstyle\sum_{i=1}^{k}\left[f_{av,b}(\tilde{a}_i,\tilde{b}_i)-f_{av,b}(\bar{a}_i,\bar{b}_i)\right]\\ &+\mu_b p(k,\tilde{b}_k,\tilde{a}_k)\end{aligned}$$

Defining $\Delta_k=\tilde{b}_k-\bar{b}_k$, taking norms, and using the triangle inequality gives

$$\begin{aligned}\Delta_{k+1}\leq \;& \Delta_1+\mu_b\textstyle\sum_{i=1}^{k}\left[L_{p,b}\|\tilde{b}_{i+1}-\tilde{b}_i\|+\chi\|\tilde{a}_{i+1}-\tilde{a}_i\|\right]+\\ &\mu_b\textstyle\sum_{i=1}^{k}\left[\gamma\|\tilde{a}_i-\bar{a}_i\|+L_{f,b}\|\tilde{b}_i-\bar{b}_i\|\right]+\|\mu_b p(k,\tilde{b}_k,\tilde{a}_k)\|\end{aligned}$$

Using (4.14) and (4.15) gives

$$\begin{aligned}\Delta_{k+1}\leq \;& \Delta_1+\mu_b\textstyle\sum_{i=1}^{k}\Big[L_{p,b}\|\mu_b f_b(\tilde{b}_i,g_a(\tilde{a}_i,x_i))+b_i^*-b_{i+1}^*\|\\ &+\chi\|\mu_a f_a(\tilde{a}_i,x_i)+a_i^*-a_{i+1}^*\|\Big]+\\ &\mu_b\textstyle\sum_{i=1}^{k}\left[\gamma\|\tilde{a}_i-\bar{a}_i\|+L_{f,b}\|\tilde{b}_i-\bar{b}_i\|\right]+\|\mu_b p(k,\tilde{b}_k,\tilde{a}_k)\|\end{aligned}$$

$$\begin{aligned}\Delta_{k+1} \leq\ & \Delta_1 + \mu_b \textstyle\sum_{i=1}^{k} \left[L_{p,b}\mu_b B_{f,b} + L_{p,b}c_b + \chi\mu_a B_f + \chi c_a\right] \\ & +\mu_b \textstyle\sum_{i=1}^{k} \left[\gamma\|\tilde{a}_i - \bar{a}_i\| + L_{f,b}\|\tilde{b}_i - \bar{b}_i\|\right] + \mu_b \\ & \|p(k,\tilde{b}_k,\tilde{a}_k) - p(k,0,\tilde{a}_k) + p(k,0,\tilde{a}_k) - p(k,0,0) + p(k,0,0)\|\end{aligned}$$

which allows us to use our assumptions to bound the last term. Using the triangle inequality yields

$$\begin{aligned}\Delta_{k+1} \leq\ & \Delta_1 + \mu_b \textstyle\sum_{i=1}^{k} \left[L_{p,b}\mu_b B_{f,b} + L_{p,b}\mu_b c_b + \chi\mu_a B_f + \chi\mu_a c_a\right] \\ & +\mu_b \textstyle\sum_{i=1}^{k} \left[\gamma w(\mu_a,T) + L_{f,b}\|\tilde{b}_i - \bar{b}_i\|\right] \\ & +\mu_b \left(\|p(k,\tilde{b}_k,\tilde{a}_k) - p(k,0,\tilde{a}_k)\| + \|p(k,0,\tilde{a}_k) - p(k,0,0)\|\right) \\ & +\mu_b\|p(k,0,0)\|\end{aligned}$$

where $w(\mu_a,T)$ is the bound on the difference between the averaged and unaveraged system trajectories in the second adaptive element provided by Theorem 5, and we have that $w(\mu_a,T) = O(\mu_a)$, given certain conditions on the speed at which the desired trajectory changes.

$$\begin{aligned}\Delta_{k+1} \leq\ & \Delta_1 + \mu_b \textstyle\sum_{i=1}^{k} \left[L_{p,b}\mu_b B_{f,b} + L_{p,b}c_b + \chi\mu_a B_f + \chi c_a\right] \\ & +\mu_b \textstyle\sum_{i=1}^{k} \left[\gamma w(\mu_a,T) + L_{f,b}\|\tilde{b}_i - \bar{b}_i\|\right] + \mu_b \left(L_{p,b}\|\tilde{b}_k\| + \chi\|\tilde{a}_k\| + B_{p,b}\right)\end{aligned}$$

We finally have

$$\begin{aligned}\Delta_{k+1} \leq\ & \Delta_1 + \mu_b k\left[L_{p,b}\mu_b(B_{f,b} + c_b) + \chi\mu_a(B_f + c_a)\right] + \mu_b k\gamma w(\mu_a,T) \\ & +\mu_b L_{f,b} \textstyle\sum_{i=1}^{k} \|\tilde{b}_i - \bar{b}_i\| + \mu_b\left(L_{p,b}h_b + \chi h_a + B_{p,b}\right)\end{aligned}$$

which, after applying the discrete Bellman-Gronwell identity[3] gives us

$$\begin{aligned}\Delta_{k+1} \leq\ & (1 + \mu_b L_{f,b})^k \left[\Delta_1 + T_b\left(L_{p,b}\mu_b B_{f,b} + L_{p,b}c_b + \chi\mu_a B_f + \chi c_a\right)\right. \\ & \left. +T_b\gamma w(\mu_a,T) + \mu_b\left(L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right]\end{aligned}$$

[3] See Appendix C.

Rearranging the terms in order to emphasize the impact of the step sizes, we have

$$\begin{aligned}\Delta_{k+1} \leq\ & (1+\mu_b L_{f,b})^k \left[\Delta_1 + \mu_b \left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. +\mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b\right]\end{aligned}$$

which gives us the bound we desire. Assuming that we start the averaged and unaveraged systems at the same place, we have

$$\begin{aligned}\Delta_{k+1} \leq\ & e^{L_{f,b}T_b} \left[\mu_b \left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. +\mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b\right] \\ & \forall k \in \{1, ..., T_b/\mu_b\}\end{aligned}$$

The last concern we must address is that we must verify that we never left the balls in which our Lipschitz continuities and bounds were valid. We must now verify that the second algorithm remains within the appropriate region. Continuing in a similar manner as in the single algorithm case, we have

$$\|\tilde{b}_k\| \leq \|\bar{b}_k\| + \Delta_k$$

Now, our assumptions about the averaged system dictate that it is exponentially stable to balls around zero error from the desired trajectory

$$\|\bar{b}_{k+1}\| \leq \beta^k \|\bar{b}_1\| + d(k, \alpha, \beta)\|\bar{a}_1\| + \frac{c_a}{(1-\alpha)(1-\beta)} + \frac{c_b}{1-\beta}$$

where

$$d(k, \alpha, \beta) = \begin{cases} \frac{\mu_b \gamma (\alpha^k - \beta^k)}{\alpha - \beta} & \alpha \neq \beta \\ \mu_b \gamma k \beta^k & \alpha = \beta \end{cases}$$

Combining this with the bound on Δ_k, and using Theorem 7 gives

$$\begin{aligned}\|\tilde{b}_k\| <\ & \beta^{k_m(\alpha,\beta)}\|\tilde{b}_1\| + d(k_m(\alpha,\beta), \alpha, \beta)\|\tilde{a}_1\| + \tfrac{c_a}{(1-\alpha)(1-\beta)} + \tfrac{c_b}{1-\beta} \\ & e^{L_{f,b}T_b} \left[\mu_b \left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. +\mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b\right]\end{aligned}$$

where, we have from Table 4.2 that

$$k_m(\alpha,\beta) = \begin{cases} \frac{\ln\left(\frac{\ln(\beta)}{\ln(\alpha)}\left(1-\frac{(\alpha-\beta)\|\tilde{b}_1\|}{\chi\|\tilde{a}_1\|}\right)\right)}{\ln(\frac{\alpha}{\beta})} & \alpha \neq \beta \\ \frac{-\|\tilde{b}_1\|}{\chi\|\tilde{a}_1\|} - \frac{1}{\ln(\beta)} & \alpha = \beta \geq \exp\left\{\frac{-\chi\|\tilde{a}_1\|}{\chi\|\tilde{a}_1\|+\|\tilde{b}_1\|}\right\} \\ 1 & \alpha = \beta < \exp\left\{\frac{-\chi\|\tilde{a}_1\|}{\chi\|\tilde{a}_1\|+\|\tilde{b}_1\|}\right\} \end{cases}$$

Thus, if the time variation is slow enough and the step sizes are small enough such that

$$\begin{aligned} h_b > \quad & \beta^{k_m(\alpha,\beta)}\|b_1\| + d(k_m(\alpha,\beta),\alpha,\beta)\|a_1\| + \tfrac{c_a}{(1-\alpha)(1-\beta)} + \tfrac{c_b}{1-\beta} \\ & e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. +\mu_a T_b \chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b} c_b\right] \end{aligned} \tag{4.22}$$

then we remain in the ball in which our assumptions hold. $\diamond$

This theorem is important not only for its quantitative information, but also for the qualitative information it provides about SFFBACs. The most important aspect we note from (4.21) is that to make the averaging error in the second element small, shrinking the step size in the second element alone (as Theorem 4 for a single adaptive element might be read to suggest) will not guarantee convergence to the averaged system. Instead we (might[4]) have to make the step size small in the first adaptive element as well to make the averaging error in the second adaptive element small.

Example 4.3 (Averaging example for SFFBACs). *Consider the series feed-foward binary adaptive compound*

$$\hat{a}_{k+1} = \hat{a}_k - \mu_a\left(\hat{a}_k - a_k^*\right)^3 \cos(\frac{2\pi k}{100})^2 - \frac{\mu_a}{4}\left(\hat{a}_k - a_k^*\right)$$

[4]The theorem provides upper bounds, but not necessarily supremums. Thus, the bounds could be quite loose.

$$\hat{b}_{k+1} = \hat{b}_k - 2\mu_b \left((\hat{b}_k - b_k^*)(1 + 3(\hat{a}_k - a_k^*)) \right) \cos(\tfrac{2\pi k}{50})^2 - \mu_b \left((\hat{b}_k - b_k^*)(1 + 3(\hat{a}_k - a_k^*)) \right)^3$$

and its associated error system

$$a_{k+1} = a_k - \mu_a a_k^3 \cos(\frac{2\pi k}{100})^2 - \frac{\mu_a}{4} a_k + a_k^* - a_{k+1}^*$$

$$b_{k+1} = b_k - 2\mu_b (b_k + 3a_k b_k) \cos(\frac{2\pi k}{50})^2 - \mu_b (b_k + 3a_k b_k)^3 + b_k^* - b_{k+1}^*$$

from which we can discern the adaptive state error functions to be

$$f_a(k, a) = -a^3 \cos(\frac{2\pi k}{100})^2 - \frac{1}{4} a$$

$$f_b(k, b, a) = -2(b + 3ab) \cos(\frac{2\pi k}{50})^2 - (b + 3ab)^3$$

We see that these system functions are associated with averaged updates

$$f_{av,a}(\bar{a}) = -\frac{1}{2}\bar{a}^3 - \frac{1}{4}\bar{a}$$

$$f_{av,b}(\bar{b}, \bar{a}) = -\left(\bar{b} + 3\bar{a}\bar{b}\right) - (\bar{b} + 3\bar{a}\bar{b})^3$$

This produces the following averaged adaptive system

$$\bar{a}_{k+1} = \bar{a}_k - \frac{\mu_a}{2}\bar{a}_k^3 - \frac{\mu_a}{4}\bar{a}_k + a_k^* - a_{k+1}^*$$

$$\bar{b}_{k+1} = \bar{b}_k - \mu_b \left(\bar{b}_k + 3\bar{a}_k\bar{b}_k\right) - \mu_b(\bar{b}_k + 3\bar{a}_k\bar{b}_k)^3 + b_k^* - b_{k+1}^*$$

Theorem 8 quantifies the difference between a trajectory that is moving in the averaged system and a trajectory moving in the unaveraged system. In order to verify its tightness for this example, we need to calculate some relevant parameters. In order to do so, let us assume our desired trajectories are

$$a_k^* = .2 \cos\left(\frac{2\pi k}{1000000}\right)$$

$$b_k^* = .3\cos\left(\frac{2\pi k}{1000000}\right)$$

We need to choose a region of interest in which to calculate uniform Lipschitz constants. We know that the averaged system needs to be uniformly contractive to the fixed point for our theorem to apply, so let's choose our area based on this restriction. Calculating the contraction constants, we have

$$\begin{aligned}\alpha &\leq \max_{\tilde{a}\in\mathcal{B}_0(h_a)} \tfrac{\|\tilde{a}-\frac{\mu_a}{2}\tilde{a}^3-\frac{\mu_a}{4}\tilde{a}\|}{\|\tilde{a}\|} \\ &\leq \max_{\tilde{a}\in\mathcal{B}_0(h_a)} \|1-\tfrac{\mu_a}{4}-\tfrac{\mu_a}{2}\tilde{a}^2\| \\ &\leq 1-\tfrac{\mu_a}{4}\end{aligned}$$

which will work (i.e. be less than one) in any ball we choose, so we can pick anything for h_a and

$$\begin{aligned}\beta &\leq \max_{\tilde{b}\in\mathcal{B}_0(h_b),\tilde{a}\in\mathcal{B}_0(h_a)} \tfrac{\|\tilde{b}-\mu_b(\tilde{b}+3\tilde{a}\tilde{b})-\mu_b(\tilde{b}+3\tilde{a}\tilde{b})^3\|}{\|\tilde{b}\|} \\ &\leq \max_{\tilde{b}\in\mathcal{B}_0(h_b),\tilde{a}\in\mathcal{B}_0(h_a)} \|1-\mu_b(1+3\tilde{a})-\mu_b\tilde{b}^2(1+3\tilde{a})^3\|\end{aligned} \tag{4.23}$$

which can be made less than one as desired if

$$\max_{\tilde{b}\in\mathcal{B}_0(h_b),\tilde{a}\in\mathcal{B}_0(h_a)} \|1-\mu_b(1+3\tilde{a})-\mu_b\tilde{b}^2(1+3\tilde{a})^3\| < 1$$

For $\mu_b = .001$, one region in which this is true is the region $a, b \in [-.2, .2]$. Thus, we choose as our region of interest $a, b \in [-.2, .2]$. Within this region (performing the calculation in (4.23) numerically) we have

$$\beta = .9996$$

$$L_{f,b} \leq \max\left\|\frac{\partial f_{av,b}}{\partial b}\right\| \leq \max\left\|-(1+3a)-3b^2(1+3a)^3\right\| = 2.0915$$

$$\gamma \leq \max\left\|\frac{\partial f_{av,b}}{\partial a}\right\| \leq \max\|-3b-9a^2b^3\| = 0.6029$$

$$\begin{aligned} L_{p,b} &\leq \max \left\| \sum_{i=1}^{k} \frac{\partial f_b(i,b,a)}{\partial b} - \frac{\partial f_{av,b}(b,a)}{\partial b} \right\| \\ &\leq \max \left\| \sum_{i=1}^{k} \frac{\partial}{\partial b} \left(-2(b+3ab)\cos(\tfrac{2\pi k}{50})^2 + (b+3ab) \right) \right\| \\ &\leq \max \left\| \sum_{i=1}^{k} \frac{\partial}{\partial b} \left(-(b+3ab)\cos(\tfrac{4\pi k}{50}) \right) \right\| \\ &\leq \max \left\| \sum_{i=1}^{k} -(1+3a)\cos(\tfrac{4\pi k}{50}) \right\| \leq 7.1704 \end{aligned}$$

$$\begin{aligned} \chi &\leq \max \left\| \sum_{i=1}^{k} \frac{\partial f_b(i,b,a)}{\partial a} - \frac{\partial f_{av,b}(b,a)}{\partial a} \right\| \\ &\leq \max \left\| \sum_{i=1}^{k} \frac{\partial}{\partial a} \left(-(b+3ab)\cos(\tfrac{4\pi k}{50}) \right) \right\| \\ &\leq \max \left\| \sum_{i=1}^{k} -3b\cos(\tfrac{4\pi k}{50}) \right\| \leq 2.6889 \end{aligned}$$

$$B_{f,b} = \max \| -2(b+3ab)\cos(\frac{2\pi k}{50})^2 - (b+3ab)^3 \| \leq 0.6728$$

$$B_{p,b} = \max \| \sum_{i=1}^{k} (0+0)\cos(\frac{4\pi i}{50}) \| \leq 0$$

$$\begin{aligned} L_p &\leq \max \left\| \sum_{i=1}^{k} \frac{\partial f_a(i,a)}{\partial a} - \frac{\partial f_{av,a}(a)}{\partial a} \right\| \\ &\leq \max \left\| \sum_{i=1}^{k} \frac{\partial}{\partial a} \left(-a^3\cos(\tfrac{2\pi k}{100})^2 + \tfrac{1}{2}a^3 \right) \right\| \\ &\leq \max \left\| \sum_{i=1}^{k} \frac{\partial}{\partial a} \frac{-1}{2} a^3 \cos(\tfrac{4\pi k}{100}) \right\| \\ &\leq \max \left\| \sum_{i=1}^{k} \frac{-3}{2} a^2 \cos(\tfrac{4\pi k}{100}) \right\| \leq 0.5078 \end{aligned}$$

$$\lambda_f \leq \max \left\| \frac{\partial f_{av,a}(a)}{\partial a} \right\| \leq \max \left\| \frac{-1}{4} - \frac{3}{2}a^2 \right\| \leq 0.31$$

$$B_f = \max \left\| \frac{-1}{4}a - a^3\cos(\frac{2\pi k}{100})^2 \right\| \leq 0.0580$$

$$B_p = \max \left\| \sum_{i=1}^{k} \frac{-1}{2} a^3 \cos(\frac{4\pi k}{100}) \right\| \leq 0.0339$$

Now, applying the theorem with a particular initialization and window lengths, T and T_b, chosen using Theorem 5, and by picking any T_b, we can create Figure 4.3.

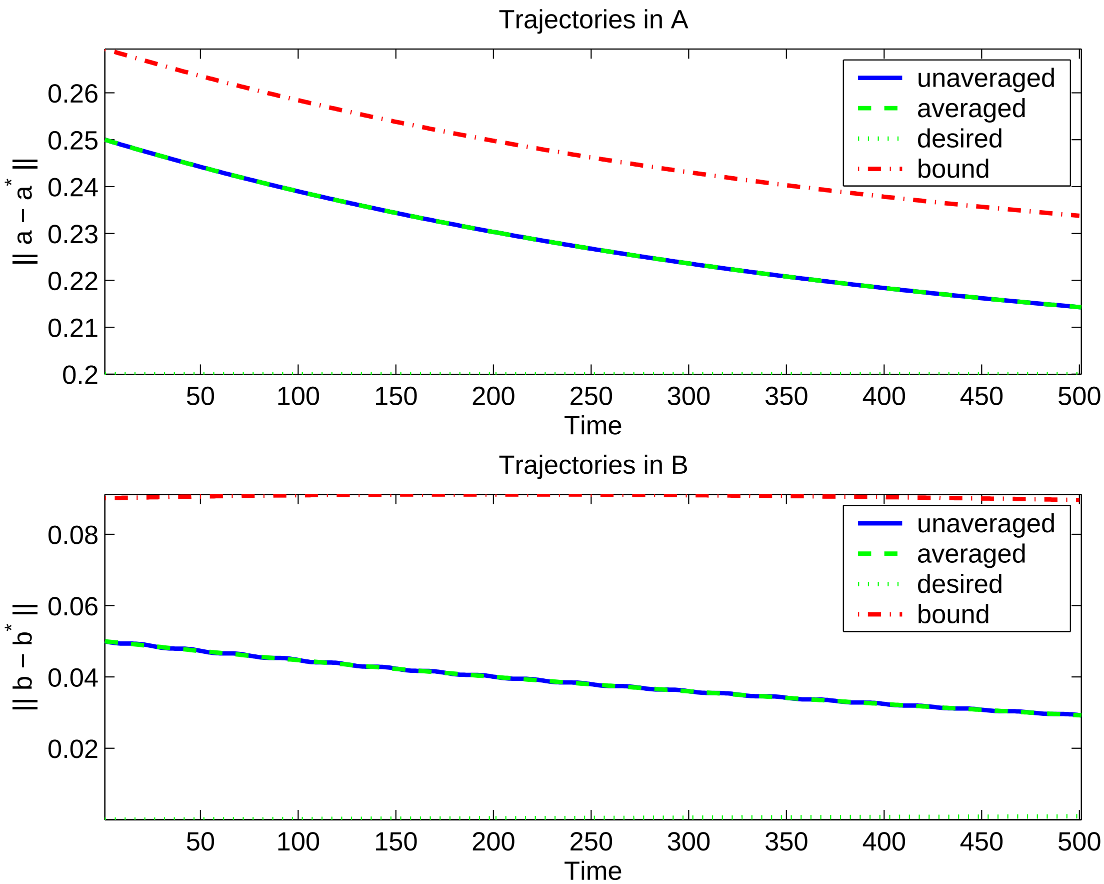

Figure 4.3: An application of the SFFDASP finite time averaging theorem. Note that by exploiting the zero mean nature of the disturbance, we have made the bound tight.

This figure indicates that the bound provided is indeed true for this case, although not at all tight.

4.1.4 SFFBAC Deterministic Hovering Theorem

Boundedness of the averaging approximation error on a finite interval only is not necessarily enough information regarding system behavior when one wishes to employ SFFBACs. Often, we want to continue using the device over an infinite amount of time, and thus, we need bounds on the error over an infinite time window. The next theorem, which is similar to theorem 5, answers this need.

Theorem 9 (SFFBAC Hovering Theorem). *Consider two adaptive elements connected together to form a series feed-forward binary adaptive compound. Assume that the first adaptive element satisfies the assumptions of Theorem 5. Furthermore, assume that the second adaptive element satisfies the assumptions of Theorem 8, with slight modifications to (4.18), (4.19), and (4.20)*

$$p_n(k,\xi,\nu) = \sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b} \left[f_b(i,\xi,g_a(\nu,x_i)) - f_{av,b}(\xi,\nu)\right]$$

$$\|p_n(k,\xi_1,\nu) - p_n(k,\xi_2,\nu)\| < L_{p,b}\|\xi_1 - \xi_2\| \quad \forall \xi_1,\xi_2 \in \mathcal{B}_0(h_b) \quad \forall \nu \in \mathcal{B}_0(h_a)$$

$$\|p_n(k,\xi,\nu_1) - p_n(k,\xi,\nu_2)\| < \chi\|\nu_1 - \nu_2\| \quad \forall \xi \in \mathcal{B}_0(h_b) \quad \forall \nu_1,\nu_2 \in \mathcal{B}_0(h_a)$$

where $\mathcal{B}_0(h_b)$ and $\mathcal{B}_0(h_a)$ are defined as in Theorem 8

$$\|p_n(k,0,0)\| \leq B_{p,b}$$

We also require

$$\begin{aligned} h_b \leq\ & \beta^{k_{max}(\alpha,\beta)}\|\bar{b}_1\| + \mu_b\gamma d(k_{max}(\alpha,\beta),\alpha,\beta)\|\bar{a}_1\| + \tfrac{c_a}{(1-\alpha)(1-\beta)} + \tfrac{c_b}{1-\beta} \\ & \tfrac{(\mu_b L_{f,b})^{T_b/\mu_b 1_{\{\mu_b > L_{f,b}^{-1}\}}})}{1-\beta^{T_b/\mu_b}} \Big[2\mu_b\gamma\alpha^{(n-1)T_b/\mu_b}u(\alpha,\beta)\|\tilde{a}_1\| + 2\mu_b\gamma\tfrac{1-\beta^{T_b/\mu_b - 1}}{1-\beta}\tfrac{c}{1-\alpha} \\ & + e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. + \mu_a T_b\chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b}c_b\right]] + \\ & e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. + \mu_a T_b\chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b}c_b\right] \end{aligned} \tag{4.24}$$

and the conditions of Theorem 5 to hold for the first adaptive element. If these assumptions hold, then over an infinite time window the difference between the averaged system's trajectory and the unaveraged system's trajectory can be bounded *by a linear function of μ_a in the first adaptive element (because Theorem 5 was assumed to hold), and by a linear function of both μ_a and μ_b in the second adaptive element*

$$
\begin{aligned}
\|\tilde{b}_k\| \leq \quad & \beta^k \|\bar{b}_1\| + \mu_b \gamma d(k,\alpha,\beta)\|\bar{a}_1\| + \tfrac{c_a}{(1-\alpha)(1-\beta)} + \tfrac{c_b}{1-\beta} \\
& \tfrac{(\mu_b L_{f,b})^{T_b/\mu_b 1_{\{\mu_b > L_{f,b}^{-1}\}}})}{1-\beta^{T_b/\mu_b}} \Big[2\mu_b \gamma \alpha^{(n-1)T_b/\mu_b} u(\alpha,\beta)\|\tilde{a}_1\| + 2\mu_b\gamma \tfrac{1-\beta^{T_b/\mu_b - 1}}{1-\beta} \tfrac{c}{1-\alpha} \\
& + e^{L_{f,b}T_b} \left[\mu_b \left(L_{p,b} B_{f,b} T_b + L_{p,b} h_b + \chi h_a + B_{p,b} \right) \right. \\
& \left. + \mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b \right]] + \\
& e^{L_{f,b}T_b} \left[\mu_b \left(L_{p,b} B_{f,b} T_b + L_{p,b} h_b + \chi h_a + B_{p,b} \right) \right. \\
& \left. + \mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b \right]
\end{aligned}
$$

$\diamond$ The trick is once again to divide the time axis in the second element up into intervals of length T_b/μ_b, and then use Theorem 8 to guarantee boundedness of the averaging error on each of these intervals. To do this, we associate with each interval a interval-averaged system trajectory, $\bar{b}_{n,k}$, $k \in \{nT_b/\mu_b, ..., (n+1)T_b/\mu_b\}$, and initialize this trajectory with the unaveraged system's parameter, $\bar{b}_{n,nT_b/\mu_b} = \tilde{b}_{nT_b/\mu_b}$. Then, we use the contraction in the averaged system's map to guarantee that each of our interval-averaged system trajectories will move toward the infinite-averaged system trajectory. Thus, we will have a relation between the interval and infinite-averaged systems, and a relation between the interval-averaged system and the unaveraged systems. We can then combine these two relations to relate the unaveraged system with the infinite-averaged system.

We begin by using (4.17) to write

$$
\begin{aligned}
\|\bar{b}_{nT_b/\mu_b} - \bar{b}_{n-1,nT_b/\mu_b}\| = \quad & \big\| \bar{b}_{nT_b/\mu_b - 1} + f_{av,b}(\bar{b}_{nT_b/\mu_b-1}, \bar{a}_{nT_b/\mu_b-1}) \\
& - \left(\bar{b}_{n-1,nT_b/\mu_b-1} + f_{av,b}(\bar{b}_{n-1,nT_b/\mu_b-1}, \bar{a}_{nT_b/\mu_b-1}) \right) \big\|
\end{aligned}
$$

Adding a clever form of zero produces

$$\begin{aligned}\|\bar{b}_{nT_b/\mu_b} - \bar{b}_{n-1,nT_b/\mu_b}\| = \; & \big\|\bar{b}_{nT_b/\mu_b-1} + \mu_b f_{av,b}(\bar{b}_{nT_b/\mu_b-1}, \bar{a}_{nT_b/\mu_b-1}) \\ & -\mu_b f_{av,b}(\bar{b}_{nT_b/\mu_b-1}, 0) + \mu_b f_{av,b}(\bar{b}_{nT_b/\mu_b-1}, 0) \\ & +\mu_b f_{av,b}(\bar{b}_{n-1,nT_b/\mu_b-1}, 0) - \mu_b f_{av,b}(\bar{b}_{n-1,nT_b/\mu_b-1}, 0) \\ & - \left(\bar{b}_{n-1,nT_b/\mu_b-1} + \mu_b f_{av,b}(\bar{b}_{n-1,nT_b/\mu_b-1}, \bar{a}_{nT_b/\mu_b-1})\right)\big\|\end{aligned}$$

Using the triangle inequality yields

$$\begin{aligned}\|\bar{b}_{nT_b/\mu_b} - \bar{b}_{n-1,nT_b/\mu_b}\| \leq \; & \big\|\bar{b}_{nT_b/\mu_b-1} + \mu_b f_{av,b}(\bar{b}_{nT_b/\mu_b-1}, 0) \\ & - \left(\bar{b}_{n-1,nT_b/\mu_b-1} + \mu_b f_{av,b}(\bar{b}_{n-1,nT_b/\mu_b-1}, 0)\right)\big\| \\ & +\|\mu_b f_{av,b}(\bar{b}_{nT_b/\mu_b-1}, \bar{a}_{nT_b/\mu_b-1}) - \mu_b f_{av,b}(\bar{b}_{nT_b/\mu_b-1}, 0)\| \\ & +\|\mu_b f_{av,b}(\bar{b}_{n-1,nT_b/\mu_b-1}, \bar{a}_{nT_b/\mu_b-1}) \\ & -\mu_b f_{av,b}(\bar{b}_{n-1,nT_b/\mu_b-1}, 0)\|\end{aligned}$$

Using the assumption about Lipschitz continuity of $f_{av,b}$ in a, we have

$$\|\bar{b}_{nT_b/\mu_b} - \bar{b}_{n-1,nT_b/\mu_b}\| \leq \beta\|\bar{b}_{nT_b/\mu_b-1} - \bar{b}_{n-1,nT_b/\mu_b-1}\| + 2\mu_b\gamma\|\bar{a}_{nT_b/\mu_b-1}\|$$

Substituting in the bound from the single algorithm hovering theorem, Theorem 5, which we know to hold for the first of the adaptive elements, we have

$$\begin{aligned}\|\bar{b}_{nT_b/\mu_b} - \bar{b}_{n-1,nT_b/\mu_b}\| \leq \; & \beta\|\bar{b}_{nT_b/\mu_b-1} - \bar{b}_{n-1,nT_b/\mu_b-1}\| + 2\mu_b\gamma\left(\alpha^{nT_b/\mu_b-2}\|a_1\|\right. \\ & \left.+\tfrac{c_a}{1-\alpha}\right)\end{aligned}$$

Running the recursion backwards until the beginning of this particular time segment and assuming $\bar{a}_1 = a_1$ yields

$$\begin{aligned}\|\bar{b}_{nT_b/\mu_b} - \bar{b}_{n-1,nT_b/\mu_b}\| \leq \; & \beta^{T_b/\mu_b}\|\bar{b}_{(n-1)T_b/\mu_b} - \bar{b}_{n-1,(n-1)T_b/\mu_b}\| + \textstyle\sum_{i=0}^{T_b/\mu_b-3}\beta^i\,(2\mu_b\gamma \\ & \left(\alpha^{nT_b/\mu_b-2-i}\|a_1\| + \tfrac{c}{1-\alpha}\right))\end{aligned}$$

Using the fact that we initialized the interval-averaged system with the unaveraged system, we have

$$\begin{aligned}\|\bar{b}_{nT_b/\mu_b} - \bar{b}_{n-1,nT_b/\mu_b}\| \leq \; & \beta^{T_b/\mu_b}\|\bar{b}_{(n-1)T_b/\mu_b} - \tilde{b}_{(n-1)T_b/\mu_b}\| + \textstyle\sum_{i=0}^{T_b/\mu_b-1}\beta^i\,(2\mu_b\gamma \\ & \left(\alpha^{nT_b/\mu_b-2-i}\|a_1\| + \tfrac{c}{1-\alpha}\right))\end{aligned}$$

Also, we know from the triangle inequality that

$$\|\bar{b}_{nT_b/\mu_b} - \tilde{b}_{nT_b/\mu_b}\| \leq \|\bar{b}_{nT_b/\mu_b} - \bar{b}_{n-1,nT_b/\mu_b}\| + \|\bar{b}_{n-1,nT_b/\mu_b} - \tilde{b}_{nT_b/\mu_b}\|$$

whose first term on the right hand side we just bounded, and whose second term on the right hand side is bounded by Theorem 8. Thus, we have

$$\begin{aligned} \|\bar{b}_{nT_b/\mu_b} - \tilde{b}_{nT_b/\mu_b}\| \leq\ & \beta^{T_b/\mu_b}\|\bar{b}_{(n-1)T_b/\mu_b} - \tilde{b}_{(n-1)T_b/\mu_b}\| + \textstyle\sum_{i=0}^{T_b/\mu_b-1} 2\mu_b\gamma\beta^i \big(\\ & \alpha^{nT_b/\mu_b-1-i}\|\tilde{a}_1\| + \tfrac{c}{1-\alpha}\big) \\ & +e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. +\mu_a T_b\chi B_f + T_b\gamma w(\mu_a, T) + T_b\chi c_a + T_b L_{p,b}c_b\right] \end{aligned}$$

Since we started the averaged an unaveraged system at the same place, we have $\|\bar{b}_{0nT_b/\mu_b} - \tilde{b}_{0nT_b/\mu_b}\| = 0$, and thus running the recursion backwards in n of the above equation yields

$$\begin{aligned} \|\bar{b}_{n\frac{T_b}{\mu_b}} - \tilde{b}_{n\frac{T_b}{\mu_b}}\| \leq\ & \tfrac{1}{1-\beta^{T_b/\mu_b}}\Big[\textstyle\sum_{i=0}^{T_b/\mu_b-1} 2\mu_b\gamma\beta^i\left(\alpha^{nT_b/\mu_b-1-i}\|\tilde{a}_1\| + \tfrac{c}{1-\alpha}\right) \\ & +e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. +\mu_a T_b\chi B_f + T_b\gamma w(\mu_a, T) + T_b\chi c_a + T_b L_{p,b}c_b\right]\Big] \end{aligned}$$

Using the sum of a finite geometric series yields

$$\begin{aligned} \|\bar{b}_{n\frac{T_b}{\mu_b}} - \tilde{b}_{n\frac{T_b}{\mu_b}}\| \leq\ & \tfrac{1}{1-\beta^{T_b/\mu_b}}\Big[2\mu_b\gamma\alpha^{(n-1)T_b/\mu_b}u(\alpha,\beta)\|\tilde{a}_1\| + 2\mu_b\gamma\tfrac{1-\beta^{T_b/\mu_b-1}}{1-\beta}\tfrac{c}{1-\alpha} \\ & +e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. +\mu_a T_b\chi B_f + T_b\gamma w(\mu_a, T) + T_b\chi c_a + T_b L_{p,b}c_b\right]\Big] \end{aligned} \tag{4.25}$$

where we recognized a term that was the same as the one we bounded in the previous theorem and called $d(k, \alpha, \beta)$, and thus defined

$$u(\alpha, \beta) = d(k_{max}(\alpha, \beta), \alpha, \beta)$$

where d and k_{max} are defined as in the previous finite time averaging theorem.

We now have a bound for the difference between the averaged and unaverged trajectories every T_b/μ_b steps. Of course, we want to bound the error between these trajectories for other times as well. To do so, we once again use the contractivity of the averaged system. Take $k \in \{nT_b/\mu_b, nT_b/\mu_b + 1, \cdots, (n+1)T_b/\mu_b\}$, then we have

$$\begin{aligned}\|\bar{b}_k - \bar{b}_{nT_b/\mu_b,k}\| &\leq \| \left(\bar{b}_{k-1} + \mu_b f_{av,b}(\bar{b}_{k-1}, \bar{a}_{k-1})\right) \\ &- \left(\bar{b}_{nT_b/\mu_b,k-1} + \mu_b f_{av,b}(\bar{b}_{nT_b/\mu_b,k-1}, \bar{a}_{k-1})\right) \| \\ &\leq \mu_b \|f_{av,b}(\bar{b}_{k-1}, \bar{a}_{k-1}) - f_{av,b}(\bar{b}_{nT_b/\mu_b,k-1}, \bar{a}_{k-1})\| \\ &\leq \mu_b L_{f,b} \|\bar{b}_{k-1} - \bar{b}_{nT_b/\mu_b,k-1}\|\end{aligned}$$

Thus, stepping back in time until we get to nT_b/μ_b gives

$$\|\bar{b}_k - \bar{b}_{nT_b/\mu_b,k}\| \leq (\mu_b L_{f,b})^{k - nT_b/\mu_b} \|\bar{b}_{nT_b/\mu_b} - \bar{b}_{nT_b/\mu_b, nT_b/\mu_b}\|$$

Thus, if $\mu_b < L_{f,b}^{-1}$, the largest difference between the interval and infinite averaged systems will be at the beginning of the interval, and if $\mu_b > L_{f,b}^{-1}$, the largest difference will be at the end of interval.

$$\begin{aligned}\|\bar{b}_k - \bar{b}_{nT_b/\mu_b,k}\| \leq \; & \tfrac{(\mu_b L_{f,b})^{T_b/\mu_b 1_{\{\mu_b > L_{f,b}^{-1}\}}})}{1-\beta^{T_b/\mu_b}} \left[2\mu_b \gamma \alpha^{(n-1)T_b/\mu_b} u(\alpha,\beta) \|\tilde{a}_1\|\right. \\ & + 2\mu_b \gamma \tfrac{1-\beta^{T_b/\mu_b - 1}}{1-\beta} \tfrac{c}{1-\alpha} \\ & + e^{L_{f,b} T_b} \left[\mu_b \left(L_{p,b} B_{f,b} T_b + L_{p,b} h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left.\left. + \mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b\right]\right]\end{aligned}$$

The triangle inequality states that

$$\|\bar{b}_k - \tilde{b}_k\| \leq \|\bar{b}_k - \bar{b}_{nT_b/\mu_b,k}\| + \|\bar{b}_{nT_b/\mu_b,k} - \tilde{b}_k\|$$

We recognize the term on the right hand side as what we bounded in the finite time averaging theorem. Furthermore, we just discussed a bound on the second term. Thus, we combine these two bounds to get

$$\begin{aligned}\|\bar{b}_k - \tilde{b}_k\| \leq \; & \tfrac{(\mu_b L_{f,b})^{T_b/\mu_b 1_{\{\mu_b > L_{f,b}^{-1}\}}}}{1-\beta^{T_b/\mu_b}} \Big[2\mu_b \gamma \alpha^{(n-1)T_b/\mu_b} u(\alpha,\beta)\|\tilde{a}_1\| + 2\mu_b\gamma \tfrac{1-\beta^{T_b/\mu_b - 1}}{1-\beta} \tfrac{c}{1-\alpha} \\ & + e^{L_{f,b}T_b} \left[\mu_b \left(L_{p,b} B_{f,b} T_b + L_{p,b} h_b + \chi h_a + B_{p,b} \right) \right. \\ & \left. + \mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b \right]] + \\ & e^{L_{f,b}T_b} \left[\mu_b \left(L_{p,b} B_{f,b} T_b + L_{p,b} h_b + \chi h_a + B_{p,b} \right) \right. \\ & \left. + \mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b \right] \end{aligned} \tag{4.26}$$

This is the bound we desired, but we must make sure that we never left the ball within which our solutions were valid. Since the averaged system is contractive

$$\|\bar{b}_{k+1}\| \leq \beta^k \|\bar{b}_1\| + \mu_b \gamma h(k,\alpha,\beta)\|\bar{a}_1\| + \frac{\mu_b \gamma c_a}{(1-\alpha)(1-\beta)} + \frac{c_b}{1-\beta} \tag{4.27}$$

Now the triangle inequality tells us that

$$\|\tilde{b}_k\| \leq \|\bar{b}_k\| + \|\bar{b}_k - \tilde{b}_k\|$$

So we have from (4.26) and (4.27)

$$\begin{aligned}\|\tilde{b}_k\| \leq \; & \beta^k \|\bar{b}_1\| + \mu_b \gamma d(k,\alpha,\beta)\|\bar{a}_1\| + \tfrac{c_a}{(1-\alpha)(1-\beta)} + \tfrac{c_b}{1-\beta} \\ & \tfrac{(\mu_b L_{f,b})^{T_b/\mu_b 1_{\{\mu_b > L_{f,b}^{-1}\}}}}{1-\beta^{T_b/\mu_b}} \Big[2\mu_b \gamma \alpha^{(n-1)T_b/\mu_b} u(\alpha,\beta)\|\tilde{a}_1\| + 2\mu_b\gamma \tfrac{1-\beta^{T_b/\mu_b - 1}}{1-\beta} \tfrac{c}{1-\alpha} \\ & + e^{L_{f,b}T_b} \left[\mu_b \left(L_{p,b} B_{f,b} T_b + L_{p,b} h_b + \chi h_a + B_{p,b} \right) \right. \\ & \left. + \mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b \right]] + \\ & e^{L_{f,b}T_b} \left[\mu_b \left(L_{p,b} B_{f,b} T_b + L_{p,b} h_b + \chi h_a + B_{p,b} \right) \right. \\ & \left. + \mu_a T_b \chi B_f + T_b \gamma w(\mu_a, T) + T_b \chi c_a + T_b L_{p,b} c_b \right] \end{aligned}$$

Recognizing that this equation behaves in k like the bound from Theorem 3 with an added constant offset, we define $k_{max}(\alpha, \beta)$ as in Theorem 3, and get that

$$\begin{aligned}\|\tilde{b}_k\| \leq \quad & \beta^{k_{max}(\alpha,\beta)}\|\bar{b}_1\| + \mu_b\gamma d(k_{max}(\alpha,\beta),\alpha,\beta)\|\bar{a}_1\| + \tfrac{c_a}{(1-\alpha)(1-\beta)} + \tfrac{c_b}{1-\beta} \\ & \tfrac{(\mu_b L_{f,b})^{T_b/\mu_b 1_{\{\mu_b > L_{f,b}^{-1}\}}})}{1-\beta^{T_b/\mu_b}} \Big[2\mu_b\gamma\alpha^{(n-1)T_b/\mu_b}u(\alpha,\beta)\|\tilde{a}_1\| + 2\mu_b\gamma\tfrac{1-\beta^{T_b/\mu_b-1}}{1-\beta}\tfrac{c}{1-\alpha} \\ & + e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. + \mu_a T_b\chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b}c_b\right]\Big] + \\ & e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. + \mu_a T_b\chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b}c_b\right]\end{aligned}$$

for any k. Thus, to remain in the balls in which our solution exists, we require

$$\begin{aligned}h_b \leq \quad & \beta^{k_{max}(\alpha,\beta)}\|\bar{b}_1\| + \mu_b\gamma d(k_{max}(\alpha,\beta),\alpha,\beta)\|\bar{a}_1\| + \tfrac{c_a}{(1-\alpha)(1-\beta)} + \tfrac{c_b}{1-\beta} \\ & \tfrac{(\mu_b L_{f,b})^{T_b/\mu_b 1_{\{\mu_b > L_{f,b}^{-1}\}}})}{1-\beta^{T_b/\mu_b}} \Big[2\mu_b\gamma\alpha^{(n-1)T_b/\mu_b}u(\alpha,\beta)\|\tilde{a}_1\| + 2\mu_b\gamma\tfrac{1-\beta^{T_b/\mu_b-1}}{1-\beta}\tfrac{c}{1-\alpha} \\ & + e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. + \mu_a T_b\chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b}c_b\right]\Big] + \\ & e^{L_{f,b}T_b}\left[\mu_b\left(L_{p,b}B_{f,b}T_b + L_{p,b}h_b + \chi h_a + B_{p,b}\right)\right. \\ & \left. + \mu_a T_b\chi B_f + T_b\gamma w(\mu_a,T) + T_b\chi c_a + T_b L_{p,b}c_b\right]\end{aligned}$$

which is the initialization accuracy required by the theorem. ⋄ We now consider an example in which we are interested in bounding the difference between the averaged and unaveraged system over an infinite interval.

Example 4.4 (SFFBAC Infinite Averaging). *We wish to use Theorem 9 in a quantitative manner for the same system that we investigated on a finite interval in Example 4.3. Noticing that we can still use all of the constants we calculated for that example, we wish to determine what length the time windows, T and T_b, should be to get the tightest bound on the difference possible. A plot showing the averaging error versus T and T_b for a particular $\|a_1\|$ and $\|b_1\|$ is shown in Figure 4.4. The plot shows the averaging error for different T in the top pane, and the*

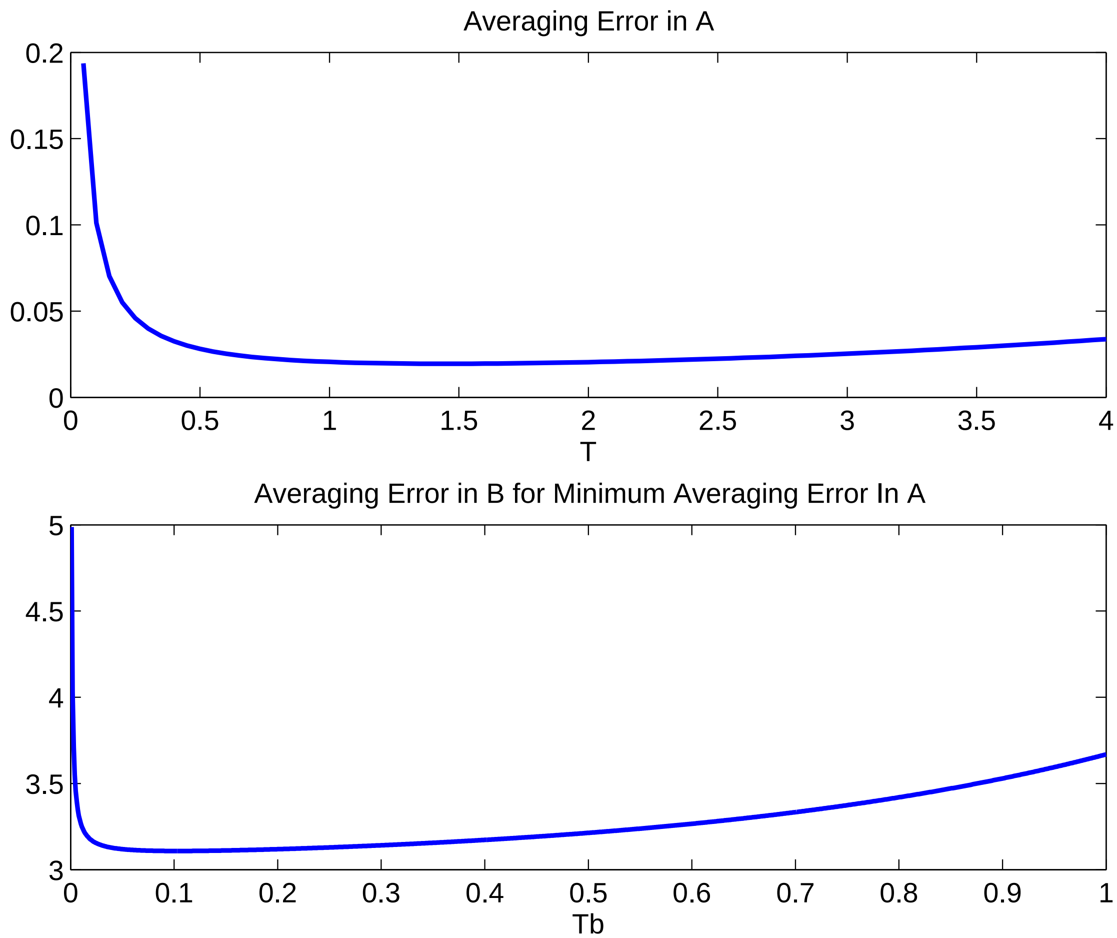

Figure 4.4: Selection of T and T_b in Example 4.4.

averaging error for different T_b, given that the optimal (associated with smallest error) T is chosen in the first adaptive element. The minimum averaging error over T and T_b is then taken, and Figure 4.5 shows that this averaging error bound is indeed correct, although terribly conservative.

It is important to notice the quantitative conservativeness of the bounds provided by this theorem, as was highlighted in the example. We derive useful qualitative information from the theorem by noting that to decrease the error between the averaged and unaveraged trajectories in the second adaptive element, one should make both μ_a and μ_b smaller. This is an interesting result, because it shows that

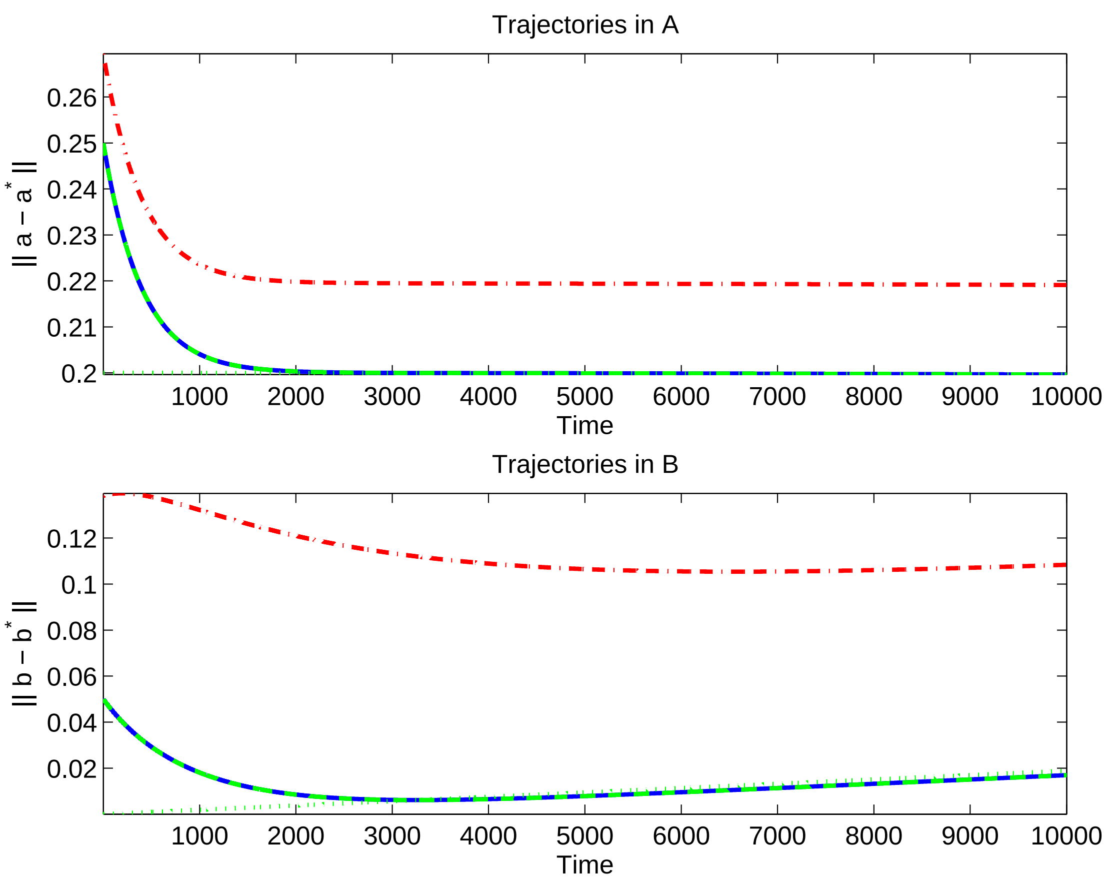

Figure 4.5: Accuracy of bound provided by Theorem 9.

in a general averaged SFFBAC designed with DaC, to guarantee that the error in the second element small, we must turn down the step size in both elements. This fact has interesting practical implications, because it argues roughly for decreasing effective step sizes as one moves along the signal path in a series feed-forward chain. Furthermore, since we needed the time variation to be on the order of the step size in each of the adaptive elements, we can draw a further conclusion. Unless the elements later in the chain are insensitive to error in the earlier elements, the effective tracking capabilities in a series feed-forward chain exploiting averaging decrease as you move forward in the chain.

4.2 Analytical Misbehavior Characterization

Now that we have given the conditions under which we can expect our SFFBAC to behave, we would like to characterize what happens when these compounds do not behave as we would expect or wish, that is, they do not converge to the desired points. While lack of convergence does not necessarily cause performance degradation or failure, if the desired trajectories were chosen to be at a maximum performance point (a likely occurrence), then the overall performance of the misbehaving system can not be better than the behaving system, and it is likely that it will be worse. One of the simplest ways to cause this to happen is to have a desired trajectory in the first element which is periodic, but too fast and large for first element to track (this can be accomplished by making the step size very small). Thus, instead of following the periodic desired trajectory, the first element stays near the average value of this trajectory. Since the first device's adaptive state is no longer convergent to its desired value, the second device's adaptation sub-element could behave non-ideally due to its dependence on proper operation of the first device. These are the ideas outlined in the next theorem.

Theorem 10 (Zero Manifold Movement in SFFBACs). *Consider a simple system built from two adaptive algorithms connected to each other in series feed-forward fashion, as in Figure 3.1*[5].

$$\hat{a}_{k+1} = \hat{a}_k + \mu f_a(\hat{a}_k - a_k^{opt})$$

$$b_{k+1} = b_k + \mu_b f_b(b_k, a_k, x_k)$$

Let f_a be a continuously differentiable function such that $f_a(0) = 0$ and $\alpha \equiv \|I + \mu_a df_a(0)\| < 1$, where $df_a(0)$ is the derivative of f_a evaluated at 0. Suppose a_k^{opt} is periodic, with an average component of a^*, such that

$$a^* = \frac{1}{N}\sum_{i=k}^{N+k} a_k^{opt} \quad \forall k$$

Define a function $b^*(k,a)$ such that

$$0 = f_b(k, b^*(k,a), a) \qquad \forall k, a \in \mathcal{D} \tag{4.28}$$

We call $b^*(k,a)$ the zero-manifold of the adaptive algorithm. Define

$$g(k,b,a) = \begin{cases} \frac{\|b+\mu_b f_b(k,b,a)-b^*(k,a)\|}{\|b-b^*(k,a)\|} & b \neq b^*(k,a) \\ \lim_{b\to b^*(k,a)} \frac{\|b+\mu_b f_b(k,b,a)-b^*(k,a)\|}{\|b-b^*(k,a)\|} & b = b^*(k,a) \end{cases}$$

and

$$\beta_{r_a,r_b} = \sup_{k\in\mathcal{N}, a\in\mathcal{B}_{a^*}(r_a),\ b\in\mathcal{B}_{b^*(k,a)}(r_b)} g(k,b,a) \tag{4.29}$$

with $\mathcal{B}_{b^*(k,a)}(r_b) = \{b \| \|b - b^*(k,a)\| \le r_b\}$ and $\mathcal{B}_{a^*}(r_a) = \{a \| \|a - a^*\| \le r_a\}$ so that we have

$$\|b + \mu_b f_b(k,b,a) - b^*(k,a)\| \le \beta_{r_a,r_b}\|b - b^*(k,a)\|$$

[5]Note that we have dropped the explicit dependance on the input output equation of the first adaptive element here by defining $f_b(k,b,a) = f_b(b, g_a(a,x))$

$$\forall a \in \mathcal{B}_{a^*}(r_a),\ b \in \mathcal{B}_{b^*(k,a)}(r_b)$$

We assume that the algorithms were chosen such that when the first element is at its optimal setting a_k^{opt} *the second adaptive element is uniformly contractive to* b_k^{opt}. *Thus* $\exists r_a h_b$ *such that* $\beta_{r_a,r_b} < 1\ \forall r_b \leq h_b$ *and* $b_k^{opt} = b^*(k, a_k^{opt})$, *as is typically done when using divide and conquer tactics. Also, assume*

$$r_a > \|a_1 - a^*\| + \frac{d}{1-\alpha} + \epsilon(\mu_a) \tag{4.30}$$

and

$$r_b > \|b_1 - b^*(a^*, 1)\| + \frac{c}{1-\beta_{r_a,r_b}} \tag{4.31}$$

where we used the following constants

$$d = \sup_{a \in \mathcal{B}_0(h_a)} \mu_a \|f_a(a_k + a^* - a_k^{opt}) - df_a(0)\left(a_k + a^* - a_k^{opt}\right)\|$$

$$c = \sup_k \|b^*(k, a^*) - b^*(k+1, a^*)\|$$

and the function, $\epsilon(\mu_a)$ *is* $O(\mu_a)$. *If these assumptions hold, the first adaptive element's state converges to a ball around* a^*, *whose size can be made arbitrarily small by shrinking* μ_a, *and the second adaptive element's state converges to a ball around* $b^*(k, a^*)$.

$$\|a_{k+1} - a^*\| \leq \alpha^k \|a_1 - a^*\| + \frac{d}{1-\alpha} + O(\mu_a)$$

$$\|b_{k+1} - b^*(k+1, a^*)\| \leq \beta_{r_a,r_b}^k \|b_1 - b^*(1, a^*)\| + \frac{c}{1-\beta_{r_a,r_b}}$$

Thus, as $\mu_a \to 0$, *it is possible that neither algorithm converges to its optimal trajectory! Instead, the first element converges to the average,* a^*, *of its optimal periodic trajectory, and the second algorithm moves around in a zero manifold,* $b^*(k, a^*)$ *(see example 4.5 for an example of a zero manifold). Since it is possible that neither adaptive element is at their optimum point, there could be performance degradation in the system.*

$\diamond$ We begin with the adaptive state equation of the first element.

$$\hat{a}_{k+1} = \hat{a}_k + \mu_a f_a(\hat{a}_k - a_k^{opt}) \tag{4.32}$$

To begin, we perform a change of variables in the two systems to make the origin the point to which we are studying stability. To do so, we define $a_k = \hat{a}_k - a^*$, and substitute it in for $\hat{a}_k$ to get

$$a_{k+1} = a_k + \mu_a f_a(a_k + a^* - a_k^{opt}) \tag{4.33}$$

Furthermore, we will base our result upon linearization, so we add a clever form of zero to get

$$a_{k+1} = a_k + \mu_a df_a(0)\left(a_k + a^* - a_k^{opt}\right) + \mu_a\left[f_a(a_k + a^* - a_k^{opt}) - df_a(0)\left(a_k + a^* - a_k^{opt}\right)\right]$$

from which we extract a term we label the "linearization error"

$$h(a,k) = f_a(a_k + a^* - a_k^{opt}) - df_a(0)\left(a_k + a^* - a_k^{opt}\right)$$

This enables us to write

$$a_{k+1} = a_k + \mu_a df_a(0)\left(a_k + a^* - a_k^{opt}\right) + \mu_a h(a_k, k)$$

We define another system which removes the effect of the difference between a^* and a_k^{opt}.

$$\bar{a}_{k+1} = \bar{a}_k + \mu_a df_a(0)\bar{a}_k + \mu_a h(\bar{a}_k, k)$$

We can use averaging theory, specifically Theorem 5, to relate these two systems. We check the applicability of this theorem by looking at the total perturbation

$$\begin{aligned} p(k,a) &= \textstyle\sum_{i=1}^{k}\left[df_a(0)\left(a + a^* - a_i^{opt}\right) + h(a,i) - \left(df_a(0) + h(a,i)\right)\right] \\ &= \textstyle\sum_{i=1}^{k} df_a(0)\left(a^* - a_i^{opt}\right) \end{aligned}$$

which, considering (4.33), must be bounded, since a_i^{opt} is periodic and real with average zero. We have just shown that assumption (2.16) of Theorem 5 holds, where we are considering our ”averaged system” to be the one dealing with $\bar{a}$, and the unaveraged one the one dealing with a. We should also check the Lipschitz continuity of the total perturbation, which is required by (4) for Theorem 5.

$$p(k,a) - p(k,a') = \sum_{i=0}^{k} df_a(0)\left(a^* - a_i^{opt}\right) - \sum_{i=0}^{k} df_a(0)\left(a^* - a_i^{opt}\right) = 0 \equiv L_p$$

Continuing on, the remaining conditions of Theorem 5 are

$$\|a_{k+1}^* - a_k^*\| \leq c \quad \forall k$$

and a condition ensuring that the system remains within the balls (of radius r_a and r_b) where our constants are valid. The first of these will hold with $c = 0$, due to the non-time varying nature of the point we are proving stability to, a^*. We will check the second condition at the end of the proof.

Assuming the Theorem applies, and we can bound the difference in the trajectories between the two systems by

$$\|\bar{a}_{k+1} - a_{k+1}\| \leq \frac{2 - \alpha^{T/\mu}}{1 - \alpha^{T/\mu}} \mu (B_p + L_p h + L_p B_f T) e^{\lambda_f T}$$

Since we know from the triangle inequality that

$$\|a_{k+1} - a^*\| \leq \|a_{k+1} - \bar{a}_{k+1}\| + \|\bar{a}_{k+1} - a^*\| \tag{4.34}$$

we now focus on verifying that the second term in the sum is bounded. To do so, we can use Theorem 1, considering $h(a_k, k)$ as a disturbance, which we uniformly bound using

$$\|h(a_k, k)\| \leq d \equiv \sup_{a \in \mathcal{B}_0(h_a)} \mu_a \|h(a,k)\|$$

Theorem 1 tells us that the second term in (4.34) is bounded by

$$\|\bar{a}_{k+1} - a^*\| \leq \alpha^k \|\bar{a}_1 - a^*\| + \frac{d}{1-\alpha}$$

Since we are considering the same initial conditions for both the averaged system and the un-averaged system, we have $\bar{a}_1 = a_1$. Using this fact, and combining our bounds for both the terms using (4.34), we have

$$\|a_{k+1} - a^*\| \leq \alpha^k \|a_1 - a^*\| + \frac{d}{1-\alpha} + \frac{2-\alpha^{T/\mu}}{1-\alpha^{T/\mu}} \mu (B_p + L_p h + L_p B_f T) e^{\lambda_f T} \tag{4.35}$$

This is our desired bound for the first element, indicating that, apart from some linearization error, the first adaptive element converges to within an ball of a^* which can be made arbitrarily small by shrinking the step size, since d is $O(\mu_a)$.

Now, let us address the part of the theorem that characterizes the behavior in the second adaptive element. To do so, we wish to quantify the distance from the zero manifold, $b^*(k, a^*)$.

$$b_{k+1} - b^*(k+1, a^*) = b_k + \mu_b f_b(k, b_k, a_k) - b^*(k, a^*) + b^*(k, a^*) - b^*(k+1, a^*)$$

using the Triangle inequality and (4.29)

$$\|b_{k+1} - b^*(k+1, a^*)\| \leq \beta_{r_a, r_b} \|b_k - b^*(k, a^*)\| + \|b^*(k, a^*) - b^*(k+1, a^*)\|$$

Defining $c = \sup_k \|b^*(k, a^*) - b^*(k+1, a^*)\|$, we have

$$\|b_{k+1} - b^*(k+1, a^*)\| \leq \beta_{r_a, r_b} \|b_k - b^*(k, a^*)\| + c$$

and running the recursion and using the sum of an infinite geometric series gives

$$\|b_{k+1} - b^*(k+1, a^*)\| \leq \beta_{r_a, r_b}^k \|b_1 - b^*(1, a^*)\| + \frac{c}{1 - \beta_{r_a, r_b}} \tag{4.36}$$

which proves that we converge to a ball around the zero-manifold, which is possibly different from the desired trajectory, $b^{opt} = b^*(k, a_k^{opt})$.

The last item which we must verify is that we remained within the balls in which our constants were valid. We do so using (4.35) and (4.36). Thus, we need

$$r_a > \|a_1 - a^*\| + \frac{d}{1-\alpha} + \frac{2-\alpha^{T/\mu}}{1-\alpha^{T/\mu}}\mu(B_p + L_p h + L_p B_f T)e^{\lambda_f T} \tag{4.37}$$

and

$$r_b > \|b_1 - b^*(a^*, 1)\| + \frac{c}{1-\beta_{r_a,r_b}} \tag{4.38}$$

for the theorem to hold. These are conditions (4.30) and (4.31) from the theorem. To see that (4.30) $\iff$ (4.37), we must show that the last term in 4.37 is $O(\mu)$ for small enough μ. This fact follows from the form we chose for $\alpha \equiv \|I + \mu df_a(0)\|$, which, for small μ and stable averaged systems is $\approx 1 - \varrho\mu$ for some constant ϱ. Given this information, we can choose T in (4.38) to guarantee the last term is $O(\mu)$. Details for this technique can be found in the hovering theorem section of [10].

To be complete, we ensure the reader is familiar with the constants used in the equations above. For Theorem 5, we used the following constants.

$$B_f = \sup_{k,\ a\in\mathcal{B}_0(h_a)} \|df_a(0)a + h(a,k)\|$$

$$\begin{aligned}\lambda_f &= \sup_{k,\ a_1,a_2\in\mathcal{B}_0(h_a)} \|\,(df_a(0)a_1 + h(a_k,k)) - (df_a(0)a_2 + h(a_k,k))\,\| \\ &\le \sup_{k,\ a_1,a_2\in\mathcal{B}_0(h_a)} \|df_a(0)\|\|a_1 - a_2\|\end{aligned}$$

Both of these exist due to differentiability of the averaged system. ◇

Example 4.5 (A Linear Misbehavior Example). *Consider the following SFF-BAC*

$$a_{k+1} = a_k - \mu_a\left(a_k - \cos(\frac{2\pi k}{100})\right) \tag{4.39}$$

$$b_{k+1} = b_k - \mu_b \left(b_k - .1 - \left(a_k - \cos\left(\frac{2\pi k}{100}\right)\right)\right) \tag{4.40}$$

Note that we have chosen to deal with linear systems here, because the results in the theorem are obtained via linearization. This is convenient, because $r_a, r_b = \infty$, and we do not have to check the conditions verifying that we remained within a ball where our contraction constants were valid, since that ball can be arbitrarily large when we have used a linear system. Continuing on, we can conclude from (4.39) and (4.40) that the optimal trajectories are

$$a_k^{opt} = \cos(\frac{2\pi k}{100})$$

$$b_k^{opt} = .1$$

Our misbehavior predicts that, for small enough μ_a, the first adaptive element will converge to a small ball around $a^* = \frac{1}{100}\sum_{i=0}^{99}\cos(\frac{2\pi k}{100}) = 0$. Also, our misbehavior theorem predicts that the second adaptive element will converge to $b^*(k, a^*)$. Using (4.28) and (4.40) the zero-manifold in the second adaptive element obeys

$$0 = b^*(k,a) - .1 - \left(a - \cos\left(\frac{2\pi k}{100}\right)\right)$$

This implies that the zero manifold in the second adaptive element is

$$b^*(k,a) = .1 + \left(a - \cos\left(\frac{2\pi k}{100}\right)\right)$$

Thus, as we make the first adaptive element's stepsize approach 0, we expect the second adaptive element to move in the manifold

$$b^*(k,a) = .1 - \cos\left(\frac{2\pi k}{100}\right)$$

Figure 4.6 shows that this is indeed the case, by plotting the trajectories of the system for several μ_a values.

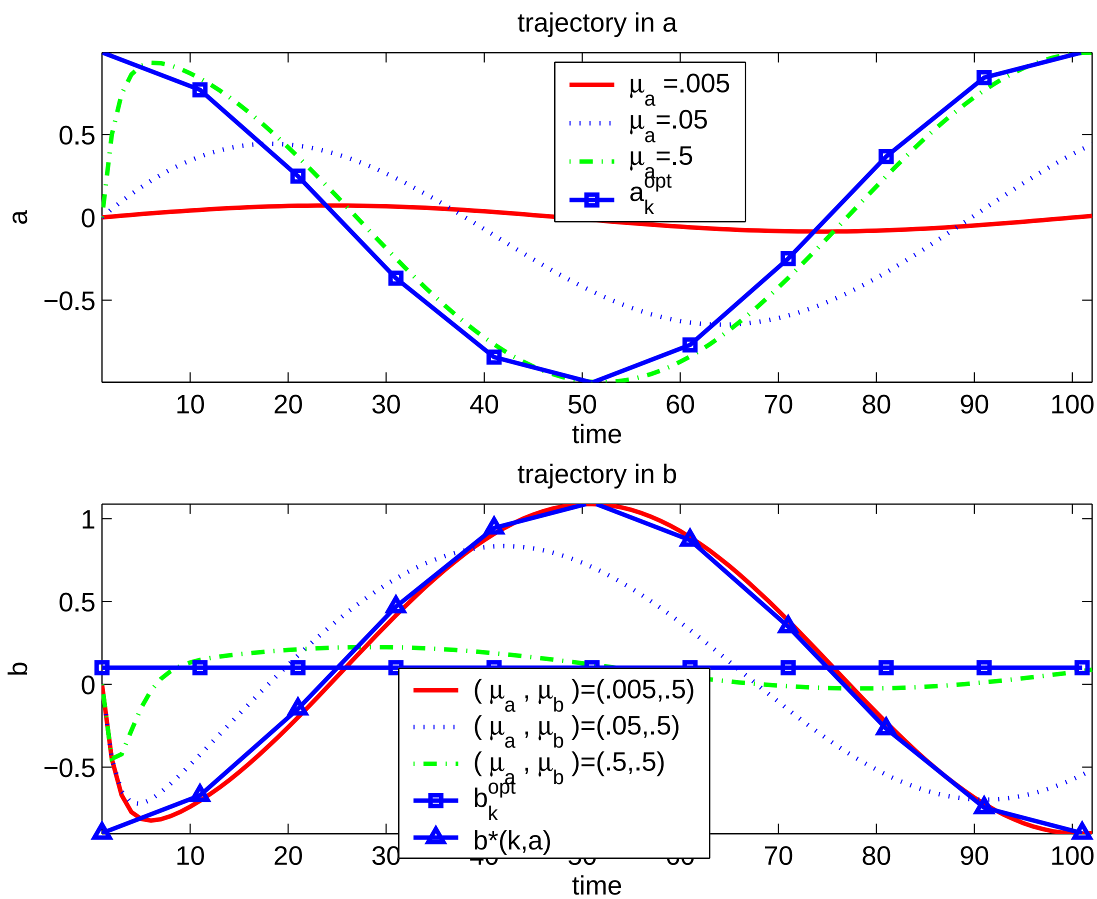

Figure 4.6: Misbehavior of two interconnected linear adaptive systems.

Example 4.6 (A Nonlinear Misbehavior Example). *To verify that the theorem holds for nonlinear systems too, we now provide a nonlinear example with*

$$a_{k+1} = a_k + \mu_a \left[(a_k - \cos(2\pi k/100)) + (a_k - \cos(2\pi k/100))^3 \right]$$

$$b_{k+1} = b_k + \mu_b \sin\left(b_k - \frac{\pi}{16} - \frac{\pi}{16} (a_k - \cos(2\pi k/100)) \right)$$

$$a_k^{opt} = \cos(2\pi k/100)$$

$$b_k^{opt} = \frac{\pi}{16}$$

$$0 = \sin\left(b^*(k, a) - \frac{\pi}{16} - \frac{\pi}{16} (a_k - \cos(2\pi k/100)) \right) \tag{4.41}$$

Since we must have both a contraction and $b^*(k, a_k^{opt}) = b_k^{opt}$, we pick a solution to (4.41) which satisfies both of these conditions

$$b^*(k, a_k) = \frac{\pi}{16} + \frac{\pi}{16} (a_k - \cos(2\pi k/100))$$

$$a^* = \frac{1}{100} \sum_{i=0}^{99} \cos(2\pi k/100) = 0$$

$$b^*(k, a^*) = \frac{\pi}{16} - \frac{\pi}{16} \cos(2\pi k/100)$$

Figure 4.7 shows thats our conclusions from the the theorem hold in this case. In particular, as you decrease the step size in the first adaptive element, it actually converges to a constant, instead of the desired periodic trajectory. This misbehavior in the first element then causes the second adaptive element to converge to a curve different from its desired trajectory. Thus, this is an example of an interconnected nonlinear adaptive system undergoing misbehavior as described in the theorem.

Thus, we have characterized one particular type of misbehavior in SFFBACs. This theorem strongly suggests that step sizes need to be chosen carefully, since decreasing them lowers the averaging error but also makes it harder for the device

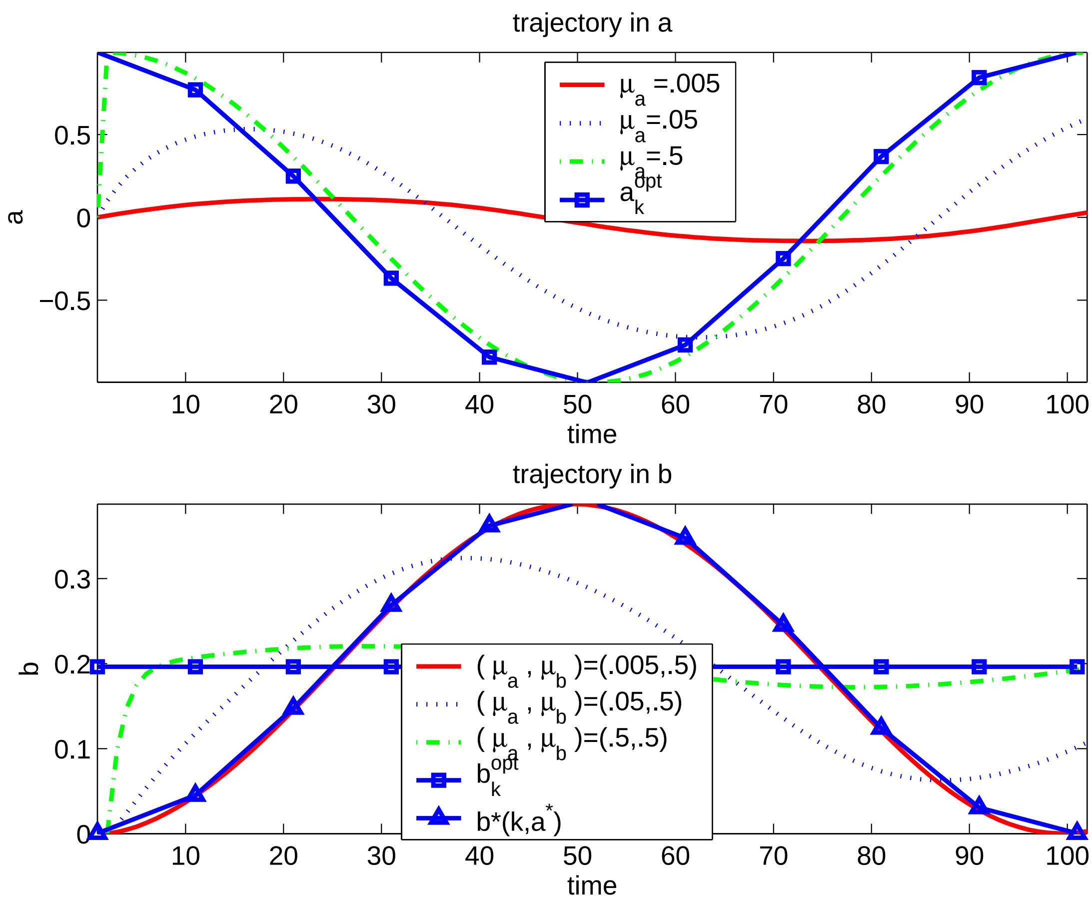

Figure 4.7: Misbehavior of two interconnected nonlinear systems.

to adapt to fast time variations. Thus, in selecting step sizes, one encounters a tradeoff between lowering the averaging error and the ability to track fast time variations.

It is important to note, however, that the results in the theorem are essentially linear. That is, nonlinear systems behave this way only to the extent that they can be approximated or bounded locally by linear systems. There are other phenomena, not mentioned in this thesis, which only nonlinear systems can have. One important phenomenon, which we will not consider here, is bifurcation. In fact, since we were interested in proving convergence within a local context, a great portion of this thesis did not address some of the fundamental aspects of nonlinear systems. This does not make the material presented in the thesis incorrect, or inaccurate, or un-useful, indeed, the theory we have provided has done mostly what it set out to do. Nevertheless, it is important to note other topics we could have discussed, and oscillation are bifurcations are a couple of these.

Chapter 5

Application of SFFBAC Theory to Simple Digital Receivers

Finally we have collected our arsenal of theorems with which to tackle a specific SFFBAC behavior/misbehavior characterization. We will now use these theorems to study the interaction between two separate adaptive elements in a digital receiver. We consider two adaptive algorithm pairs, an automatic gain control followed by carrier recovery, and timing recovery followed by equalization.

5.1 Application: Behavior of Gain Control and Phase Recovery

We now apply our conditions and theorem to a simple digital receiver architecture.

5.1.1 The System Architecture

Consider a simple QPSK digital communications system as depicted in Figure 5.1. The source symbols a_k, which are chosen in an independently and identically distributed manner from a QPSK constellation with unit power, are passed through

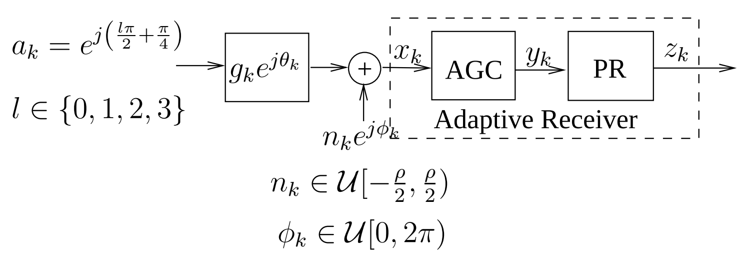

Figure 5.1: A Simple QPSK Digital Communications System.

a channel which scales, phase shifts, and adds i.i.d. zero-mean noise that is independent of the source symbols. From the diagram, we can discern that the output of the channel, x_k can be written as

$$x_k = g_k e^{j\theta_k} a_k + n_k e^{j\phi_k}$$

The receiver then processes this input with the intention of keeping the power delivered to the rest of the receiver approximately constant. To perform this task, the receiver employs an automatic gain control, which forms its output y_k by scaling its input x_k by its adaptive estimate of the optimal gain, $\hat{g}_k$ in

$$y_k = \hat{g}_k x_k = \hat{g}_k g_k e^{j\theta_k} + \hat{g}_k n_k e^{j\phi_k}$$

The receiver then must compensate for the phase shift θ_k which occurred in the channel. To do so, it uses a phase recovery adaptive device, which de-rotates its input with its phase estimate, $\hat{\theta}_k$ to produce as output

$$z_k = y_k e^{-j\hat{\theta}_k} = \hat{g}_k g_k e^{j(\theta_k - \hat{\theta}_k)} + \hat{g}_k n_k e^{j(\phi_k - \hat{\theta}_k)}$$

5.1.2 The Adaptive Algorithms

There are a number of adaptive algorithms which the receiver can employ in order to chose the processing gain, $\hat{g}_k$, and phase, $\hat{\theta}_k$. We will consider the following automatic gain control algorithm [11]

$$\hat{g}_{k+1} = \hat{g}_k + \mu_{GC} \left(1 - \hat{g}_k^2 |x_k|^2\right) \text{sign}[g_k] \tag{5.1}$$

We will also consider two possible choices for the phase recovery algorithm, a blind algorithm and a trained algorithm. The trained algorithm exploits (unusual) knowledge at the receiver of the transmitted symbols, a_k, and is commonly referred to as a (trained) PLL [12]. It obeys the recursion

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \mu_{PR}\mathrm{Im}\left\{a_k^* y_k e^{-j\hat{\theta}_k}\right\} \tag{5.2}$$

The blind algorithm [13] we will consider for phase recovery exploits the QPSK constellation by using a 4th power nonlinearity in

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \mu_{PR}\mathrm{Im}\left\{\left(x_k e^{-j\hat{\theta}_k}\right)^4\right\} \tag{5.3}$$

5.1.3 Averaged Behavior

We now consider the averaged behavior of the two algorithms given our input by calculating the expectation of the adaptive state functions (i.e. the expected value of the portions of (5.1), (5.2), and (5.3) following μ). For the automatic gain control we have

$$\begin{aligned} S_g(\bar{g}_k) &= \mathsf{E}\left[1 - \bar{g}_k g_k \left|a_k e^{j\theta_k} + n_k e^{j\phi_k}\right|^2\right] \\ &= 1 - (\bar{g}_k g_k + \bar{g}_k \mathsf{E}[|n_k|^2]) \end{aligned}$$

For the trained PLL we have

$$\begin{aligned} S_\theta(\bar{g}, \bar{\theta}) &= \mathsf{E}\left[\mathrm{Im}\left\{a_k^* g_k \bar{g}_k a_k e^{j(\theta_k - \bar{\theta}_k)}\right\}\right] \\ &= \bar{g}_k g_k \sin\left(\theta_k - \bar{\theta}_k\right) \end{aligned}$$

Finally, the blind fourth power phase recovery unit has the following expected update

$$\begin{aligned} S_\theta(\bar{g}, \hat{\theta}) &= \mathsf{E}\left[\mathrm{Im}\left\{\left(g_k \bar{g}_k e^{j\theta_k} a_k e^{-j\theta}\right)^4\right\}\right] \\ &= g_k^4 \bar{g}_k^4 \sin\left(4(\theta_k - \bar{\theta}_k)\right) \end{aligned}$$

5.1.4 Average Contractive Fixed Points

Conditions 3 and 4 require the system to be contractive on average to the fixed points of interest. To investigate these conditions, we now consider the fixed points of the averaged system, which are the points where the update, S, we calculated above is equal to zero. These are the points where the adaptive element's state would stay if it reached them. Calculating the fixed point for the automatic gain control, we have

$$S_g(g_k^*) = 0 \;\Rightarrow\; g_k^* = \frac{1}{g_k + \mathsf{E}[|n|^2]}$$

For the PLL, the fixed point, θ_k^*, which zeros the average update is

$$S_\theta(\bar{g}, \theta_k^*) = 0 \;\Rightarrow\; \theta_k^* = \theta_k + m\pi \text{ or } \bar{g} = 0$$

Finally, the 4th power phase recovery has fixed points

$$S_\theta(\bar{g}, \theta_k^*) = 0 \;\Rightarrow\; \theta_k^* = \theta_k + \frac{m\pi}{4},\; m \in \mathcal{N}$$

Next we determine which fixed points are contractive, because the algorithm will move to such points on average if it starts near enough to them. For each stationary point we calculate a contractivity constant, and check if it is < 1 for some region surrounding the fixed point of interest. For the gain control, the contractivity constant is written as

$$\alpha(r_a) = \sup_{(k,\bar{g}_k)\in\mathcal{B}} \frac{|\bar{g}_k + \mu_{GC}\left(1 - \bar{g}_k\left(g_k + \mathsf{E}[|n_k|^2]\right)\right) - g_k^*|}{|\bar{g}_k - g_k^*|}$$

where

$$\mathcal{B} = \{k \in \mathcal{N},\; g_k|\, |g_k - g_k^*| < r_a\}$$

Similarly, the contractivity constant for a fixed point, θ_k^*, of the Costas loop is

$$\beta_c(r_b) = \sup_{(k,\bar{\theta}_k)\in\mathcal{C}} \frac{|\bar{\theta}_k + \mu_\theta \bar{g}_k g_k \sin\left(\theta_k - \bar{\theta}_k\right) - \theta_k^*|}{|\theta_k^* - \bar{\theta}_k|}$$

and the contractivity constant for the 4th power phase recovery is

$$\beta_4(r_b) = \sup_{(k,\bar{\theta}_k)\in\mathcal{C}} \frac{|\bar{\theta}_k + \mu_\theta \bar{g}_k^4 g_k^4 \sin\left(4\left(\theta_k - \bar{\theta}_k\right)\right) - \theta_k^*|}{|\theta_k^* - \bar{\theta}_k|}$$

where, for both the Costas and the 4th power, the set which the supremum is taken over is

$$\mathcal{C} = \left\{ k \in \mathcal{N},\ \bar{\theta}_k |\ |\bar{\theta}_k - \theta_k^*| < r_a \right\}$$

For the correct minima, $\theta_k^* = \theta_k$, we have for the Costas loop

$$\beta_c(r_b) \leq 1 - \mu_\theta \bar{g}_k g_k \text{sinc}(r_b) \quad r_b < \frac{\pi}{2} \tag{5.4}$$

and for the fourth power, we have

$$\beta_4(r_b) \leq 1 - 4\mu_\theta \bar{g}_k^4 g_k^4 \text{sinc}(4r_b) \quad r_b < \frac{\pi}{4} \tag{5.5}$$

where we have defined $\text{sinc}(x) = \frac{\sin(x)}{x}$. This satisfies Conditions 3 and 4. Now all that remains is to check condition 5, and then the remaining conditions will stipulate the channels for which the receiver will work well.

5.1.5 Comparison Using Sensitivities

Next we investigate condition 5, and what it, together with Theorem 7, implies about the comparative behavior of the two phase recovery algorithms we are considering. In particular, we wish to compare the two possible phase recovery schemes based on their sensitivity to misadjustment in the gain (a concept similar to that in [22]). Using the fact that the mean value theorem allow us to use the norm of the derivative to determine the Lipschitz constant we have

$$\begin{aligned} \chi_{costas} &\leq \sup_{\bar{\theta} \in \mathcal{C}, \bar{g} \in \mathcal{B}} \left\| \frac{\partial S_\theta(g,\theta)}{\partial g} \right\| \\ &\leq \sup_{\bar{\theta} \in \mathcal{C}, \bar{g} \in \mathcal{B}} \left\| g_k \bar{g}_k \sin\left(\theta_k - \bar{\theta}_k\right) \right\| \end{aligned}$$

Similarly, we can take the derivative with respect to the gain of the Fourth Power nonlinearity based carrier recovery scheme

$$\chi_{4th} \leq \sup_{\bar{\theta} \in \mathcal{C}, \bar{g} \in \mathcal{B}} \left\| 4\bar{g}_k^3 g_k^4 \sin\left(4\left(\theta_k - \bar{\theta}_k\right)\right) \right\|$$

For the correct minima, $\theta_k^* = \theta_k$, we have for the Costas loop

$$\chi_{costas} \leq \begin{cases} \mu_\theta g_k \sin(r_b) & r_b < \frac{\pi}{2} \\ \mu_\theta g_k & \pi > r_b > \frac{\pi}{2} \end{cases} \tag{5.6}$$

and for the fourth power phase recovery

$$\chi_{4^{th}} \leq \begin{cases} \mu_\theta 4 \bar{g}_k^3 g_k^4 \sin(4r_b) & r_b < \frac{\pi}{8} \\ \mu_\theta 4 \bar{g}_k^3 g_k^4 & \frac{\pi}{4} < r_b < \frac{\pi}{8} \end{cases} \tag{5.7}$$

We can now use the sensitivities and the contraction constants we have calculated to compare the bounds on the error in the averaged system for the two phase recovery algorithms. Specifically, subject to some conditions, our theorem bounds the error in the phase recovery algorithm due to misadjustment in the gain algorithm by a constant proportional to $\frac{\chi}{1-\beta}$. Thus, we compare $\frac{\chi}{1-\beta}$ for the two phase algorithms of interest. Using (5.4) and (5.6), we have for the Costas loop

$$\frac{\chi}{1-\beta} = \frac{\mu_\theta g_k \sin(r_b)}{\mu_\theta \bar{g}_k g_k \mathrm{sinc}(r_b)} = \frac{r_b}{\bar{g}_k} \quad r_b < \frac{\pi}{4}$$

and using (5.5) and (5.7), we have for the fourth power phase recovery

$$\frac{\chi}{1-\beta} = \frac{\mu_\theta 4 \bar{g}_k^3 g_k^4 \sin(4r_b)}{4\mu_\theta \bar{g}_k g_k \mathrm{sinc}(4r_b)} = \frac{4r_b}{\bar{g}_k} \quad r_b < \frac{\pi}{8}$$

Thus, for $r_b < \frac{\pi}{8}$, the 4th power phase algorithm is 4 times as sensitive as the Costas loop to errors in the gain algorithm. This suggests that the Costas loop is superior to the fourth power loop due to its higher robustness to errors in the gain tracking algorithm. This comparison is not quite fair, however, because the Costas loop needs to know the source symbols to operate, which on an infinite horizon, is unlikely in communications systems, since the entire point is to transmit information unknown to the receiver. Thus, the training segment will appear only intermittently. Thus, the extra jitter (along with the extra stationary points of the blind 4th power scheme relative to the trained Costas loop) can be viewed as the penalty paid for removing the need for sending training data, or data known to the receiver. Since the theorem using χ provides only an upper bound on the adaptive state error, there is no guarantee that the upper bound will be tight. Thus, we are led to wonder if the sensitivity we calculated actually can correspond to a larger error (instead of only the possibility of a larger error). Figure 5.2 indicates that this is indeed the case, since from it we can ascertain that the fluctuations in the adaptive gain are causing noisy behavior in the 4th power nonlinearity based carrier recovery scheme.

5.2 Timing Recovery and Equalization

We now wish to use the theorems we have developed to give us some insight into the way a system employing timing recovery followed by equalization will behave. The system which we are considering is diagrammed in Figure 5.3.

One expects that the timing algorithm, which is in front of the equalizer, will affect the equalizer's adaptation. We will use our theorems and some simulations to confirm that this is indeed the case and to quantify their interaction. We will use our observations to draw some conclusions as to how to choose good timing and equalization algorithms and step sizes.

5.2.1 Exact Gradient Stability

We have shown in Theorem 6 that if the timing algorithm is exponentially stable to a "good" location, the equalization algorithm is exponentially stable to a "good" location when the timing algorithm is frozen at its "good" location, and the equalization element is continuous with respect to the timing phase, then when

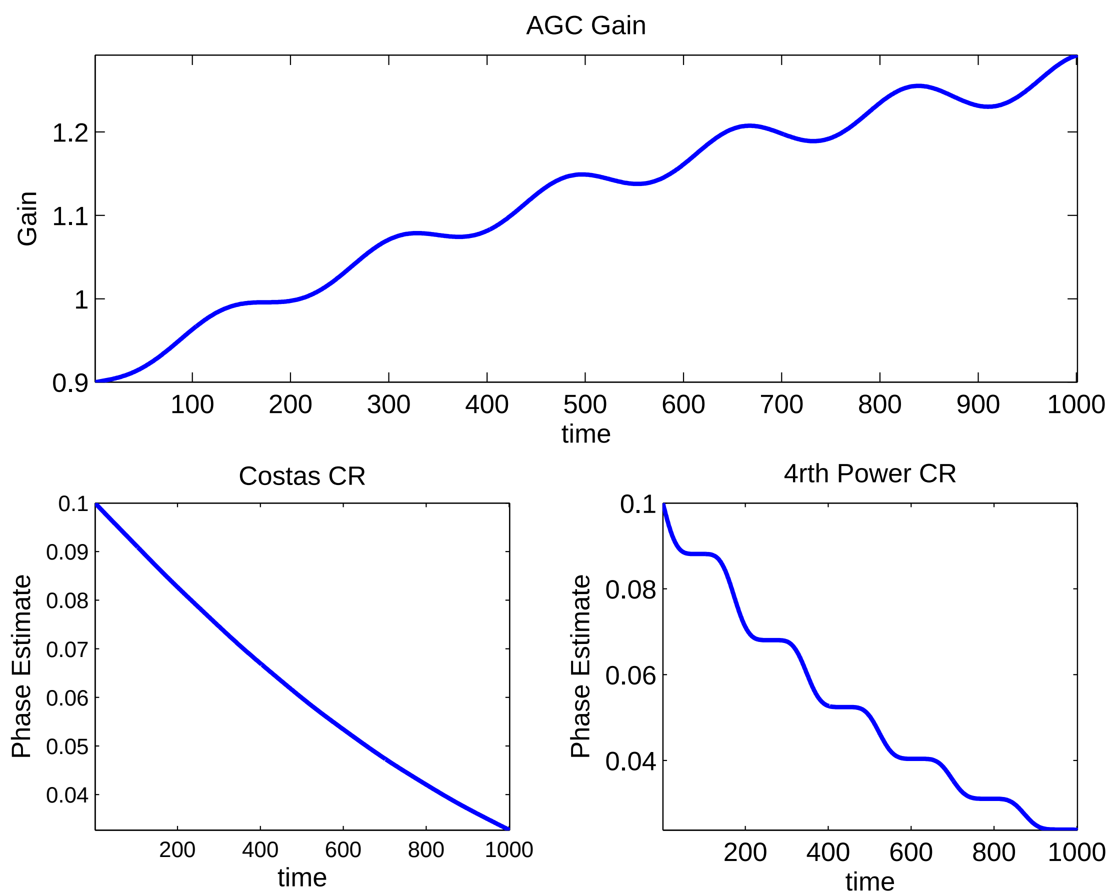

Figure 5.2: A sample adaptation showing the greater sensitivity of the 4th power CR to the AGC gain.

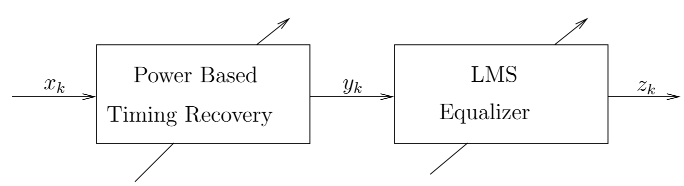

Figure 5.3: A SFFBAC found in digital receivers: Timing Recovery followed by Equalization.

hooked together, both algorithms will be exponentially stable to their "good" locations. Let's see what Theorem 6 can tell us about how a power maximization timing scheme followed by a least mean squares equalizer will behave. Since it is for a disturbance free adaptive algorithm, to apply this theorem we consider the noiseless, exact gradient case. Looking at Tables 2.3 and 2.5, we see that the sensitivities are

$$S_{\tau|g,\mathbf{f}}(\tau|g_k,\mathbf{f}) = |g_k|^2 \mathsf{E}[|a_n|^2] \sum_n \mathrm{Re}\left\{h^*_{\tau,\mathbf{f}}[k,n] dh_{\tau,\mathbf{f}}[k,n]\right\} + \sigma^2$$

with

$$h_{\hat{\tau},\mathbf{f}}[k,n] = \sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-l)T + \hat{\tau}_k) p((k-l-n)T + \hat{\tau}_k - \tau_i) f_l$$

for the power maximization timing recovery, and

$$S_{\mathbf{f}|\tau,g,\theta}(\mathbf{f}|\tau,g,\hat{\theta}) = \hat{g}_{k-m} e^{j\hat{\theta}_{k-m}} \mathsf{E}[|a|^2] h^*_{\hat{\tau}}[k-m,k-\delta] - \textstyle\sum_{l=0}^{P-1} \hat{g}_{k-m}\hat{g}_{k-l}$$
$$e^{j(\hat{\theta}_{k-m}-\hat{\theta}_{k-l})} \textstyle\sum_n \mathsf{E}[|a|^2] h^*_{\hat{\tau}}[k-m,n] h_{\hat{\tau}}[k-l,n] f_l + \sum_{l=0}^{P-1} \mathsf{E}[v^*_{k-m} v_{k-l}] f_l$$

with

$$h_\tau[k,n] = \sum_{i=0}^{P-1} c_i(kT + \hat{\tau}_k) p((k-n)T + \hat{\tau}_k - \tau_i)$$

for the least mean squares equalizer. Since we have chosen to use a series feedforward form, the output of the timing recovery is fed into the input of the equalizer (see Figure 5.3). Thus, in the sensitivity for the timing from Table 2.3, we must remove the equalizer. Furthermore, since there will be no automatic gain control or phase recovery in our analysis, we must remove these from the sensitivity as well. To do so, we can rewrite $h_{\hat{\tau},\mathbf{f}}[k,n]$ from the last entry in Table 2.3 as

$$h_\tau[k,n] = \sum_{i=1}^{Q} c_i(kT + \hat{\tau}_k) p((k-n)T + \hat{\tau}_k - \tau_i(kT + \hat{\tau}_k)) \tag{5.8}$$

for the time varying channel case, and

$$h_\tau[k,n] = \sum_{i=1}^{Q} c_i p((k-n)T + \hat{\tau}_k - \tau_i)$$

for the time-invariant channel case. The exact gradient descent for the power based timing algorithm is then

$$\hat{\tau}_{k+1} = \hat{\tau}_k + \mu_a \mathsf{E}[|a_n|^2] \sum_n \mathrm{Re}\left\{h_\tau^*[k,n] dh_\tau[k,n]\right\}$$

Continuing on to the equalizer, and remembering that we are investigating the noiseless case, the expected update for the mth equalizer tap is

$$f_{k+1,m} = f_{k,m} + \mu_b \mathsf{E}\left[|a|^2\right] \left(h_\tau^*[k-m, k-\delta] - \sum_n h_\tau^*[k-m,n] \sum_{l=0}^{P-1} h_\tau[k-l,n] f_{k,l} \right)$$

We now wish to establish whether or not the conditions of Theorem 6 hold in this case. Define $\hat{\tau}_k^*$ to be a solution to

$$\sum_n \mathrm{Re}\left\{ h_{\hat{\tau}_k^*}^*[k,n] dh_{\hat{\tau}_k^*}[k,n] \right\} = 0$$

To find a contraction constant (if it exists) for this stationary point, we must look at

$$\alpha_{r_a} = \sup_{\{\tau \mid \|\tau - \hat{\tau}_k^*\| < r_a\}} \frac{\|\tau + \mu_a \sum_n \mathrm{Re}\left\{h_\tau^*[k,n] dh_\tau[k,n]\right\} - \hat{\tau}_k^*\|}{\|\tau - \hat{\tau}_k^*\|} \tag{5.9}$$

Let's consider the particular case of a time-invariant channel,

$$c(\tau) = \delta(\tau - 3.1) - 0.2\delta(\tau + 1.2T) \tag{5.10}$$

which is plotted in the upper left pane of Figure 5.4. For this example, we use a raised cosine pulse shape with roll-off factor 0.25, a BPSK source signal, and no additive channel noise. For this case, we can use numerical techniques to calculate the stationary points and to test for contraction constants for each stationary point. In the upper right pane of Figure 5.4, we plot the sensitivity function for various values of the adaptive parameter. From this plot, we can determine the stationary points, since they will be the values of τ that zero the drawn curve. Then, we plot the function that we took the supremum of in (5.9) for each stationary point

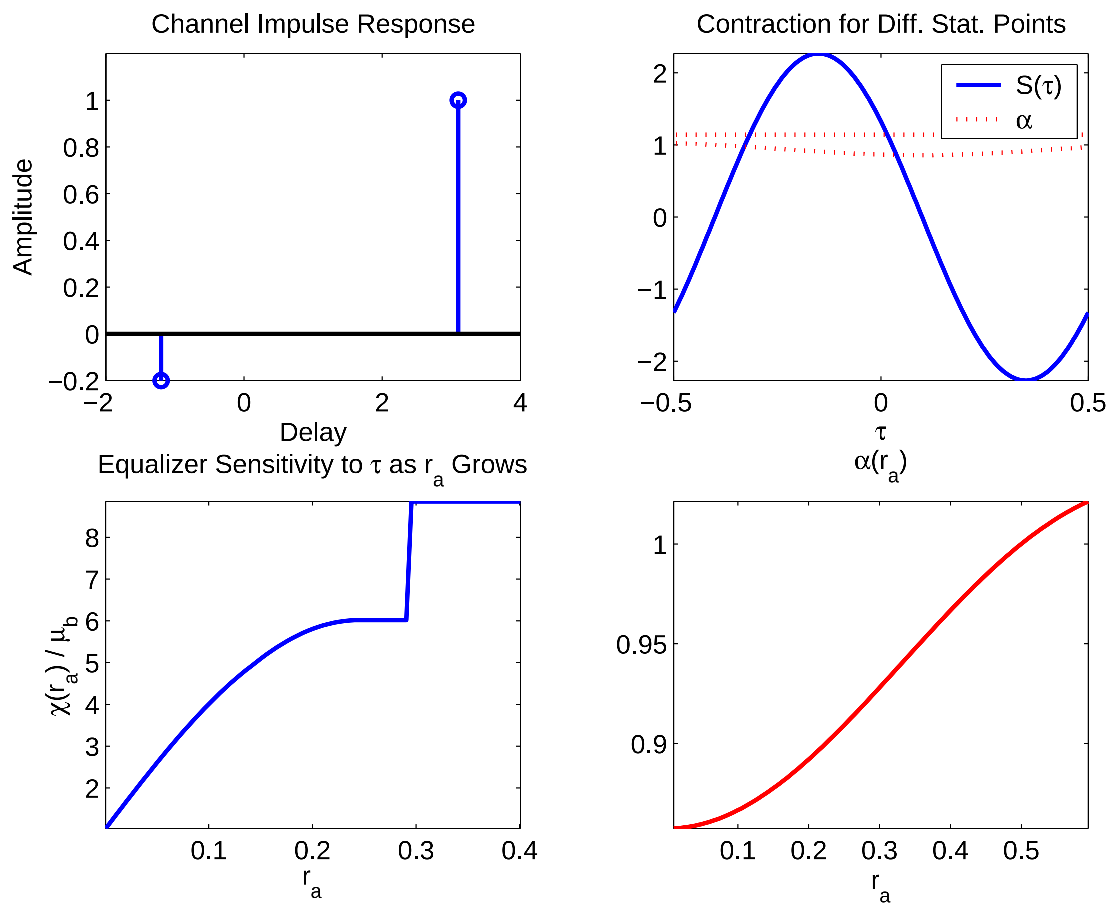

Figure 5.4: The constants involved in applying Theorem 6 to a timing equalizer system.

in order to determine if this stationary point has a contraction towards it (these are the dotted lines). Any fixed point that has a region surrounding it in which its dotted line lies below one is a locally stable fixed point. Note the "u-shaped" nature of the lower dotted line, which indicates the contraction constant grows as we move away from the minimum in the timing algorithm.

To apply Theorem 6, we also need to investigate whether or not we have a contraction in the equalizer (the second element) when the timing (first) element is at its stationary point, and we need to determine the Lipschitz constant which indicates the sensitivity of the equalization algorithm to the timing parameter. The first of these involves looking at

$$\beta_{r_b} = \max_{\mathbf{f}|\|\mathbf{f}-\mathbf{f}_k^*\|<r_b} \frac{\|\mathbf{f} + \mu_b S_{\mathbf{f}}(\tau_k^*)\|}{\|\mathbf{f} - \mathbf{f}_k^*\|}$$

where $\mathbf{f}_k^*$ solves

$$S_{\mathbf{f}|\tau}(\mathbf{f}_k^*|\tau^*) = 0 \tag{5.11}$$

To find the solution to this equation in a compact manner, introduce the $P \times P$ matrix R_h such that

$$[R_h]_{i,j} = \sum_n h_{\hat{\tau}}^*[k-i,n] h_{\hat{\tau}}[k-j,n]$$

Also introduce the $1 \times P$ vector $\mathbf{P}$ such that

$$[\mathbf{P}]_i = h_{\tau^*}[k-i, k-\delta]$$

Then, the LMS sensitivity can be written as

$$S_\tau(\mathbf{f}) = \mathsf{E}[|a|^2] \left(\mathbf{P} - R_h \mathbf{f}\right) \tag{5.12}$$

Assuming R_h is invertible, we can solve (5.11) for $\mathbf{f}_k^*$, getting

$$\mathbf{f}_k^* = R_h^{-1}\mathbf{P} \tag{5.13}$$

Recall that we are interested in finding the contraction constants. Thus, we are interested in relating $\|\mathbf{f} - \mathbf{f}_k^*\|$ with $\|\mathbf{f} + \mu_b \mathsf{E}[|a|^2] \left(\mathbf{P} - R_h \mathbf{f}\right) - \mathbf{f}_k^*\|$. Rearranging terms in the second expression and substituting in (5.13) gives

$$\begin{aligned} \|\mathbf{f} - \mathbf{f}_k^* + \mu_b \mathsf{E}[|a|^2] R_h \left(R_h^{-1}\mathbf{P} - \mathbf{f}\right)\| &= \|\mathbf{f} - \mathbf{f}_k^* - \mu_b \mathsf{E}[|a|^2] R_h \left(\mathbf{f} - \mathbf{f}_k^*\right)\| \\ &= \|\left(I - \mu_b \mathsf{E}[|a|^2] R_h\right)\left(\mathbf{f} - \mathbf{f}_k^*\right)\| \end{aligned}$$

or

$$\|\mathbf{f} - \mathbf{f}_k^* + \mu_b \mathsf{E}[|a|^2] R_h \left(R_h^{-1}\mathbf{P} - \mathbf{f}\right)\| \le \|I - \mu_b \mathsf{E}[|a|^2] R_h\| \|\mathbf{f} - \mathbf{f}_k^*\|$$

where the reader is expected to be able to deal with the sloppy notation.[1] Thus, we see that will have a contraction if

$$\beta = \|I - \mu_b \mathsf{E}[|a|^2]R_h\| < 1$$

This is a condition we can easily check. This equation also shows for LMS, that if we have a uniform[2] contraction, then it is a global uniform contraction, since R_h is the same for every equalizer, $\mathbf{f}$. This is a special property of LMS[3], which leads to its nice global convergence under mild conditions. We can determine both the stationary point and the update using numerical methods too. For the particular example channel we are investigating and our particular step size and timing recovery timing algorithm, a two tap real LMS equalizer has a contraction to the optimal location given by (5.10) with rate $\beta = .9985$.

Finally, to determine the Lipschitz constant that indicates the equalization algorithm's sensitivity to the timing parameter, we look at

$$\chi_{r_a} = \sup_{\xi_1,\xi_2 \in \mathcal{B}_{\hat{\tau}_k^*}} \frac{|\mu_b| \|S_{\mathbf{f}|\tau}(\mathbf{f}|\xi_1) - S_{\mathbf{f}|\tau}(\mathbf{f}|\xi_2)\|}{\|\xi_1 - \xi_2\|}$$

Substituting in (5.12), we have

$$\chi = \sup_{\tau_1,\tau_2} \frac{|\mu_b| \|\mathsf{E}[|a|^2](\mathbf{P}_{\tau 1} - R_{h,\tau_1}\mathbf{f}) - \mathsf{E}[|a|^2](\mathbf{P}_{\tau_2} - R_{h,\tau_2}\mathbf{f})\|}{\|\tau_1 - \tau_2\|}$$

$$\chi \le \sup_{\tau_1,\tau_2} \frac{|\mu_b| \mathsf{E}[|a|^2] \left(\|\mathbf{P}_{\tau_1} - \mathbf{P}_{\tau_1}\| + \|R_{h,\tau_1} - R_{h,\tau_2}\| \|\mathbf{f}\|\right)}{\|\tau_1 - \tau_2\|}$$

which is a form that is good for numerical calculations. This is a function of τ_1 and τ_2 which we must take the supremum over within the ball of size r_a. This Lipschitz constant, is shown as a function of the size of the ball, r_a, which we

[1] There is an linear operator norm implied here. It has the usual definition: $\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$

[2] remember that R_h depends on the current time, k

[3] and other algorithms with linear updates (quadratic costs) in the parameters.

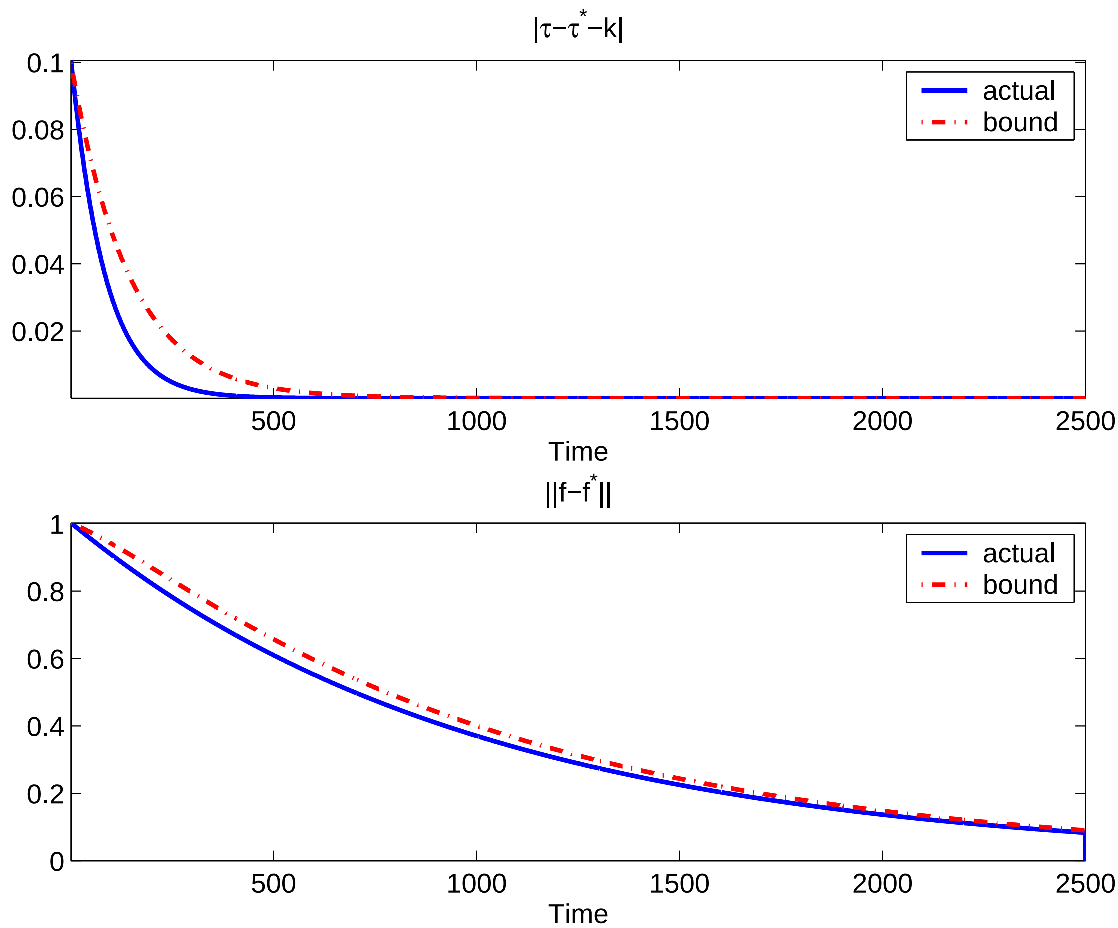

Figure 5.5: Theorem 6 applied to a digital receiver containing an equalizer and a timing recovery unit undergoing exact gradient descent.

restrict ourselves to in the bottom left pane of Figure 5.4. Furthermore, Figure 5.5 shows that the bound provided by Theorem 6 is indeed working.

5.2.2 Inexact Gradient Descent

However, since they are often stochastic gradient descents of some cost function, many adaptive equalization and timing algorithms are not exponentially stable: usually, only their averaged system has such a property. Thus, we wish to use averaging theory to quantify what sort of deviations from the averaged trajectory we could have. Continuing with the two algorithms and channel example we had previously, we now use the stochastic gradient descent form of the two algorithms. Referring to Table 2.2, for the power based timing recovery the unaveraged adaptive state equations becomes

$$\hat{\tau}_{k+1} = \hat{\tau}_k + \mu_a \mathrm{Re}\left\{y^*(kT + \hat{\tau}_k) dy(kT + \hat{\tau}_k)\right\}$$

and, referring to Table 2.4, the unaveraged adaptive state equation for the LMS equalizer is

$$\hat{\mathbf{f}}_{k+1} = \hat{\mathbf{f}}_k + \mu_a \left(a_{k-\delta} - \mathbf{r}_k^T \hat{\mathbf{f}}_k\right) \mathbf{r}_k^*$$

where

$$\begin{aligned} y(kT + \hat{\tau}_k) = \quad & \hat{g}_k e^{-j\hat{\theta}_k} \textstyle\sum_n a_n \sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-l)T + \hat{\tau}_k) \\ & p((k-l-n)T + \hat{\tau}_k - \tau_i) f_l + v_k \end{aligned} \tag{5.14}$$

and

$$\mathbf{r}_k = \left[\hat{g}_m e^{-j\hat{\theta}_m} \sum_n a_n \sum_{i=1}^{Q} c_i(mT + \hat{\tau}_m) p((m-n)T + \hat{\tau}_m - \tau_i) + v_m\right]_{m \in [k:-1:k-P+1]} \tag{5.15}$$

Once again considering that our receiver will contain only timing recovery followed by equalization, (5.14) and (5.15) become

$$y(kT + \hat{\tau}_k) = \sum_n a_n \sum_{i=1}^{Q} c_i(kT + \hat{\tau}_k) p((k-n)T + \hat{\tau}_k - \tau_i) + v_k \tag{5.16}$$

$$\mathbf{r}_k = \left[\sum_n a_n \sum_{i=1}^{Q} c_i(mT + \hat{\tau}_m) p((m-n)T + \hat{\tau}_m - \tau_i) + v_m\right]_{m \in [k:-1:k-P+1]}$$

Let us further simplify the model by removing the noise, v_k.

$$\hat{\tau}_{k+1} = \hat{\tau}_k + \mu_a \mathsf{E}[|a_n|^2] \sum_n \mathrm{Re}\left\{h_\tau^*[k,n] dh_\tau[k,n]\right\} + \mu_a \mathsf{E}[|v_k|^2] \sum_\zeta q_{\hat{\tau}}[k,\zeta] dq_{\hat{\tau}}[k,\zeta]$$

$$\hat{f}_{k+1,m} = \hat{f}_{k,m} + \mu_b \mathsf{E}\left[|a|^2\right] \Big(h^*_\tau[k-m,k-\delta] - \textstyle\sum_n h^*_\tau[k-m,n] \sum_{l=0}^{P-1} h_\tau[k-l,n] \hat{f}_{k,l} \Big) + \mu_b E[|v_k|^2] \sum_{l=0}^{P-1} \sum_\zeta q_{\hat{\tau}}[k-m,\zeta] q^*_{\hat{\tau}}[k-l,\zeta] \hat{f}_{k,l}$$

We wish to use the SFFBAC Hovering Theorem on this binary adaptive compound. To do so, we must calculate some parameters and verify some assumptions. The parameters which we must calculate are

- **The averaged error system and unaveraged error system for the timing recovery**

$$f_{av}(a) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(i,a,x_i)$$

To get this, we need to choose a desired trajectory. Operating under the assumption that we designed the adaptive element by stochastic gradient techniques, as is the case with power based timing recovery, we choose the stationary points of the averaged system to be the desired points. Thus, the trajectory $\hat{\tau}^*_k$ solves

$$\mathsf{E}[|a_n|^2] \sum_n \mathrm{Re}\left\{ h^*_{\hat{\tau}^*_k}[k,n] dh_{\hat{\tau}^*_k}[k,n] \right\} + \mu_a \mathsf{E}[|v_k|^2] \sum_\zeta q_{\hat{\tau}^*_k}[k,\zeta] dq_{\hat{\tau}^*_k}[k,\zeta] = 0$$

as with the case of the disturbance free timing equalizer system, we will resort to numerical techniques to calculate this, since it is analytically messy and difficult to solve when the received signal contains more than one multipath component. Once we have determined, $\hat{\tau}^*_k$ numerically, the averaged error system becomes

$$\begin{aligned} \bar{\tau}_{k+1} = \; & \bar{\tau}_k + \mu_a \mathsf{E}[|a_n|^2] \textstyle\sum_n \mathrm{Re}\left\{ h^*_{\bar{\tau}_k+\hat{\tau}^*_k}[k,n] dh_{\bar{\tau}_k+\hat{\tau}^*_k}[k,n] \right\} \\ & + \mu_a \mathsf{E}[|v_k|^2] \textstyle\sum_\zeta q_{\bar{\tau}_k+\hat{\tau}^*_k}[k,\zeta] dq_{\bar{\tau}_k+\hat{\tau}^*_k}[k,\zeta] + \hat{\tau}^*_k - \hat{\tau}^*_{k+1} \end{aligned} \tag{5.17}$$

and the unaveraged error system becomes

$$\tau_{k+1} = \tau_k + \mu_a \mathrm{Re}\left\{ y^*(kT+\tau_k+\hat{\tau}^*_k) dy(kT+\tau_k+\hat{\tau}^*_k) \right\} + \hat{\tau}^*_k - \hat{\tau}^*_{k+1} \tag{5.18}$$

- **The averaged error system and unaveraged error system for equalization**

$$f_{av,b}(b,a) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f_b(i,b,a)$$

To get the averaged error system for the equalizer, we use the averaged system and a desired trajectory. Once again, we proceed assuming that the equalizer has been designed with a stochastic gradient descent mindset, as is the case with LMS. In this case, the averaged system's fixed points determine the desired trajectory, and $\mathbf{f}_k^*$ solves

$$\Big(\mu_b \mathsf{E}\left[|a|^2\right] \Big(h_\tau^*[k-m,k-\delta] - \textstyle\sum_n h_\tau^*[k-m,n] \sum_{l=0}^{P-1} h_\tau[k-l,n]\hat{f}_{k,l}\Big)$$
$$+\mu_b E[|v_k|^2] \textstyle\sum_{l=0}^{P-1} \sum_\zeta q_{\hat{\tau}}[k-m,\zeta] q_{\hat{\tau}}^*[k-l,\zeta]\hat{f}_{k,l}\Big) = 0$$

Proceeding as we did in the disturbance free case, we rewrite this system in a matrix/vector description. First, we define a $P \times 1$ column vector, $\mathbf{P}_{\tau_k,k}$, and a $P \times P$ matrix, $R_{\tau_k,k}$ such that

$$\left[\mathbf{P}_{\tau_k,k}\right]_i = \mathsf{E}\left[|a|^2\right] h_{\tau^*}[k-i,k-\delta] \tag{5.19}$$

$$\left[R_{\tau_k,k}\right]_{i,j} = \begin{array}{c} \mathsf{E}\left[|a|^2\right] \sum_n h_{\tau+\hat{\tau}}^*[k-i,n] h_{\tau+\hat{\tau}}[k-j,n] \\ +E[|v_k|^2] \sum_\zeta q_{\tau+\hat{\tau}}[k-i,\zeta] q_{\tau+\hat{\tau}}^*[k-j,\zeta] \end{array} \tag{5.20}$$

to make the averaged system

$$\hat{\mathbf{f}}_{k+1} = \hat{\mathbf{f}}_k + \mu_b \left(\mathbf{P}_{\tau_k,k} - R_{\tau_k,k}\hat{\mathbf{f}}_k\right)$$

Recalling that we want the desired equalizer to be the stationary point of the averaged equalizer system when the timing recovery is at its optimal point, $\mathbf{f}_k^*$ solves

$$\mathbf{f}_k^* = R_{\hat{\tau}_k^*,k}^{-1} \mathbf{P}_{\hat{\tau}_k^*,k} \tag{5.21}$$

Using this as our desired trajectory gives an averaged equalizer error system

$$\bar{\mathbf{f}}_{k+1} = \bar{\mathbf{f}}_k + \mu_b \left(\mathbf{P}_{\tau_k,k} - R_{\tau_k,k}\left(\bar{\mathbf{f}}_k + \mathbf{f}_k^*\right)\right) + \mathbf{f}_k^* - \mathbf{f}_{k+1}^* \tag{5.22}$$

and an instantaneous equalizer error system

$$\mathbf{f}_{k+1} = \mathbf{f}_k + \mu_b \left(a_{k-\delta} - \mathbf{r}_k^T (\mathbf{f}_k + \mathbf{f}_k^*) \right) \mathbf{r}_k^* + \mathbf{f}_k^* - \mathbf{f}_{k+1}^* \tag{5.23}$$

- **Boundedness of time variation of the desired timing signal**.

$$\|\hat{\tau}_{k+1}^* - \hat{\tau}_k^*\| \leq c_a \quad \forall k$$

c_a is a constant which we will have to find numerically after we solve for $\hat{\tau}_k^*$.

- **Boundedness of time variation of the desired equalizer**.

$$\|\mathbf{f}_{k+1}^* - \mathbf{f}_k^*\| \leq c_b$$

c_b is a constant which we will calculate numerically using (5.21).

- **Contraction of the timing adaptive element to zero error**

$$\|\bar{a}_k + \mu_a f_{av}(\bar{a}_k)\| \leq \alpha \|\bar{a}_k\|$$

We will find α numerically, and show how it grows as the ball of interest around $\hat{\tau}_k^*$ grows.

- **Contraction of the equalizer to zero error, given perfect timing**

$$\|\bar{b}_k + \mu_b f_{av,b}(\bar{b}_k, 0)\| \leq \beta \|\bar{b}_k\|$$

Using (5.22), we have

$$\beta = \max_{k, \|\mathbf{f}\| < r_b} \frac{\|\mathbf{f}_k + \mu_b \left(\mathbf{P}_{\hat{\tau}_k^*, k} - R_{\hat{\tau}_k^*, k} \left(\mathbf{f}_k + \mathbf{f}_k^* \right) \right) \|}{\|\mathbf{f}\|}$$

substituting in (5.21) gives

$$\beta = \max_{k, \|\mathbf{f}\| < r_b} \frac{\|\mathbf{f}_k - \mu_b R_{\hat{\tau}_k^*, k} \mathbf{f}_k\|}{\|\mathbf{f}\|} \leq \max_k \|I - \mu_b R_{\hat{\tau}_k^*, k}\|$$

- **Total averaging perturbation for the timing recovery**

$$p_{nT/\mu}(k,a)=\sum_{i=nT/\mu}^{nT/\mu+k}[f(i,a,x_i)-f_{av}(a)]$$

To get the total perturbation for the timing recovery, we subtract (5.17) and (5.18) to get

$$\begin{aligned}p_{nT/\mu}(k,\tau)= & \textstyle\sum_{i=nT/\mu}^{nT/\mu+k}\left[\mathrm{Re}\left\{y^*(kT+\tau+\hat{\tau}_k^*)dy(kT+\tau+\hat{\tau}_k^*)\right\}\right. \\ & -\mathsf{E}[|a_n|^2]\textstyle\sum_n \mathrm{Re}\left\{h^*_{\tau+\hat{\tau}_k^*}[k,n]dh_{\tau+\hat{\tau}_k^*}[k,n]\right\} \\ & \left.-\mathsf{E}[|v_k|^2]\textstyle\sum_\zeta q_{\tau+\hat{\tau}_k^*}[k,\zeta]dq_{\tau+\hat{\tau}_k^*}[k,\zeta]\right]\end{aligned}$$

Using (5.16) we know that

$$y(kT+\tau+\hat{\tau}_k^*)=\sum_n a_n h_{\tau+\hat{\tau}_k^*}[k,n]+v_k$$

where

$$h_{\tau+\hat{\tau}_k^*}[k,n]=\sum_{i=1}^{Q}c_i(kT+\tau+\hat{\tau}_k^*)p((k-n)T+\tau+\hat{\tau}_k^*-\tau_i)$$

so we can write

$$\begin{aligned}p_{nT/\mu}(k,\tau)= & \textstyle\sum_{i=nT/\mu}^{nT/\mu+k}\left[\sum_{n_1}\sum_{n_2}\mathrm{Re}\left\{a^*_{n_1}a_{n_2}h^*_{\tau+\hat{\tau}_k^*}[k,n_1]dh_{\tau+\hat{\tau}_k^*}[k,n_2]\right.\right. \\ & \left.+v^*_{n_1}v_{n_2}q^*_{\tau+\hat{\tau}_k^*}[k,n_1]dq_{\tau+\hat{\tau}_k^*}[k,n_2]\right\} \\ & -\mathsf{E}[|a_n|^2]\textstyle\sum_n \mathrm{Re}\left\{h^*_{\tau+\hat{\tau}_k^*}[k,n]dh_{\tau+\hat{\tau}_k^*}[k,n]\right\} \\ & \left.-\mathsf{E}[|v_k|^2]\textstyle\sum_\zeta q_{\tau+\hat{\tau}_k^*}[k,\zeta]dq_{\tau+\hat{\tau}_k^*}[k,\zeta]\right]\end{aligned}$$

Rearranging yields

$$\begin{aligned}p_{nT/\mu}(k,\tau)= & \textstyle\sum_{i=nT/\mu}^{nT/\mu+k}\sum_{n_1}\sum_{n_2}\left[\mathrm{Re}\left\{\left(a^*_{n_1}a_{n_2}-\mathsf{E}[|a_n|^2]\delta[n_1-n_2]\right)\right.\right. \\ & h^*_{\tau+\hat{\tau}_k^*}[k,n_1]dh_{\tau+\hat{\tau}_k^*}[k,n_2]+\left(v^*_{n_1}v_{n_2}-\mathsf{E}[|v_k|^2]\delta[n_1-n_2]\right) \\ & \left.\left.q^*_{\tau+\hat{\tau}_k^*}[k,n_1]dq_{\tau+\hat{\tau}_k^*}[k,n_2]\right\}\right]\end{aligned}$$

which is the total perturbation for the timing recovery averaging approximation.

- **Lipschitz continuity for the total perturbation of the timing recovery**

$$\|p_{nT/\mu}(k,\tau_1) - p_{nT/\mu}(k,\tau_2)\| \leq \lambda_p \|\tau_1 - \tau_2\|$$

Beginning with the calculation, we have

$$\begin{aligned}\lambda_p = \quad & \max_{k,\tau_1,\tau_2 \in B_0(r_a)} \textstyle\sum_{i=nT/\mu}^{nT/\mu+k} \sum_{n_1} \sum_{n_2} \frac{1}{\|\tau_1-\tau_2\|} \Big[\text{Re} \Big\{ \big(a_{n_1}^* a_{n_2} \\ & - \mathsf{E}[|a_n|^2]\delta[n_1 - n_2] \big) \Big(h^*_{\tau_1+\hat{\tau}_k^*}[k,n_1] dh_{\tau_1+\hat{\tau}_k^*}[k,n_2] \\ & - h^*_{\tau_2+\hat{\tau}_k^*}[k,n_1] dh_{\tau_2+\hat{\tau}_k^*}[k,n_2] \Big) + \big(v_{n_1}^* v_{n_2} - \mathsf{E}[|v_k|^2]\delta[n_1 - n_2] \big) \\ & \Big(q^*_{\tau_1+\hat{\tau}_k^*}[k,n_1] dq_{\tau_1+\hat{\tau}_k^*}[k,n_2] q^*_{\tau_2+\hat{\tau}_k^*}[k,n_1] dq_{\tau_2+\hat{\tau}_k^*}[k,n_2] \Big) \Big\} \Big]\end{aligned}$$

Introduce the following symbols $a_{max} = \max_{k,j} a_k^* a_j$ and $v_{max} = \max_{k,j} v_k^* v_j$. Then we have

$$\begin{aligned}\lambda_p \leq \quad & \max_{k,\tau_1,\tau_2 \in B_0(r_a)} \textstyle\sum_{i=nT/\mu}^{nT/\mu+k} \sum_{n_1} \sum_{n_2} \frac{1}{\|\tau_1-\tau_2\|} \big[(a_{max} - \mathsf{E}[|a_n|^2]\delta[n_1 - n_2]) \\ & \Big\| h^*_{\tau_1+\hat{\tau}_k^*}[k,n_1] dh_{\tau_1+\hat{\tau}_k^*}[k,n_2] - h^*_{\tau_2+\hat{\tau}_k^*}[k,n_1] dh_{\tau_2+\hat{\tau}_k^*}[k,n_2] \Big\| \\ & + (v_{max} - \mathsf{E}[|v_k|^2]\delta[n_1 - n_2]) \Big\| q^*_{\tau_1+\hat{\tau}_k^*}[k,n_1] dq_{\tau_1+\hat{\tau}_k^*}[k,n_2] \\ & q^*_{\tau_2+\hat{\tau}_k^*}[k,n_1] dq_{\tau_2+\hat{\tau}_k^*}[k,n_2] \Big\| \big]\end{aligned}$$

which is the formula we will use to numerically bound this Lipschitz constant.

- **Total averaging perturbation for the equalizer**

$$p_n(k,b,a) = \sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b} [f_b(i,b,g_a(a,x_i)) - f_{av,b}(b,a)]$$

Subtracting the update functions in (5.22) and (5.23) gives

$$p_n(k,\mathbf{f},\tau) = \sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b} \left[(\mathbf{P}_{\tau,i} - R_{\tau,i}(\mathbf{f} + \mathbf{f}_i^*)) - \left(a_{i-\delta} - \mathbf{r}_{\tau,i}^T (\mathbf{f} + \mathbf{f}_i^*) \right) \mathbf{r}_{\tau,i}^* \right]$$

- **Lipschitz continuity of the total perturbation to changes in the equalizer**

$$\|p_n(k,\mathbf{f}_1,\tau) - p_n(k,\mathbf{f}_2,\tau)\| < L_{p,b}\|\mathbf{f}_1 - \mathbf{f}_2\| \quad \forall \mathbf{f}_1,\mathbf{f}_2 \in \mathcal{B}_0(h_b) \quad \forall \tau \in \mathcal{B}_0(h_a)$$

In order to simplify matters, let us assume differentiability of the adaptive state equation, since the example we are dealing with is differentiable. In this case, we can write

$$L_{p,b} = \max_{\mathbf{f}\in\mathcal{B}_0(h_b)} \left\| \sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b} \frac{\partial \left[(\mathbf{P}_{\tau,i} - R_{\tau,i}\left(\mathbf{f}+\mathbf{f}_i^*\right)) - \left(a_{i-\delta} - \mathbf{r}_{\tau,i}^T(\mathbf{f}+\mathbf{f}_i^*)\right)\mathbf{r}_{\tau,i}^* \right]}{\partial \mathbf{f}} \right\|$$

which, using some properties of matrix differentiation, becomes

$$L_{p,b} = \max_{\mathbf{f}\in\mathcal{B}_0(h_b)} \left\| \sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b} (R_{\tau,i} - A_i) \right\|$$

where we have introduced the $P \times P$ matrix A whose elements are $[A]_{m,n} = r_{k-m}r_{k-n}$. Note that this gives us an easier form with which to calculate the Lipschitz constant numerically.

- **Lipschitz continuity of the total perturbation in the equalizer to changes in timing**

$$\|p_n(k,\mathbf{f},\tau_1) - p_n(k,\mathbf{f},\tau_2)\| < \chi\|\tau_1 - \tau_2\| \quad \forall \mathbf{f} \in \mathcal{B}_0(h_b) \quad \forall \tau_1,\tau_2 \in \mathcal{B}_0(h_a)$$

assuming differentiability with respect to τ, which we have in the example we are considering, we can write

$$\chi < \left\| \sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b} \frac{\partial \mathbf{P}_{\tau,i}}{\partial \tau} - \frac{\partial R_{\tau,i}}{\partial \tau}(\mathbf{f}+\mathbf{f}_i^*) - a_{i-\delta}\frac{\partial \mathbf{r}_{\tau,i}^*}{\partial \tau} + \frac{\partial \mathbf{r}_{\tau,i}^T(\mathbf{f}+\mathbf{f}_i^*)\mathbf{r}_{\tau,i}^*}{\partial \tau} \right\|$$

Recalling (5.19) we know

$$\left[\frac{\partial \mathbf{P}_{\tau_k,k}}{\partial \tau}\right]_i = \mathsf{E}\left[|a|^2\right] dh_{\tau+\tau^*}[k-i,k-\delta] \tag{5.24}$$

and recalling (5.20) we know

$$\begin{aligned} \left[\frac{\partial R_{\tau_k,k}}{\partial \tau}\right]_{i,j} = \quad & \mathsf{E}\left[|a|^2\right] \textstyle\sum_n \left(dh^*_{\tau+\hat{\tau}}[k-i,n]h_{\tau+\hat{\tau}}[k-j,n]\right. \\ & \left.+h^*_{\tau+\hat{\tau}}[k-i,n]dh_{\tau+\hat{\tau}}[k-j,n]\right) \\ & +E[|v_k|^2] \textstyle\sum_\zeta \left(dq_{\tau+\hat{\tau}}[k-i,\zeta]q^*_{\tau+\hat{\tau}}[k-j,\zeta]\right. \\ & \left.+q_{\tau+\hat{\tau}}[k-i,\zeta]dq^*_{\tau+\hat{\tau}}[k-j,\zeta]\right) \end{aligned} \tag{5.25}$$

and using some elementary properties of matrix differentiation, we have

$$\begin{aligned}\frac{\partial \mathbf{r}_{\tau,i}^T(\mathbf{f}+\mathbf{f}_i^*)\mathbf{r}_{\tau,i}^*}{\partial\tau} &= \frac{\partial \mathbf{r}_{\tau,i}^T}{\partial\tau}(\mathbf{f}+\mathbf{f}_i^*)\mathbf{r}_{\tau,i}^* + \mathbf{r}_{\tau,i}^T(\mathbf{f}+\mathbf{f}_i^*)\frac{\partial \mathbf{r}_{\tau,i}^*}{\partial\tau} \\ &= \left(\mathbf{r}_{\tau,i}^*\frac{\partial \mathbf{r}_{\tau,i}^T}{\partial\tau} + \frac{\partial \mathbf{r}_{\tau,i}^*}{\partial\tau}\mathbf{r}_{\tau,i}^T\right)(\mathbf{f}+\mathbf{f}_i^*)\end{aligned}$$

where

$$\left[\frac{\partial \mathbf{r}_{\tau,i}}{\partial\tau}\right]_{m\in[k:-1:k-P+1]} = \sum_n a_n dh_{\tau+\tau_m^*}[m,n] + \sum_i v_i dq_{\tau+\tau_m^*}[m,i]$$

We now have all of the terms which we need to calculate the Lipschitz constant. We shall calculate them numerically for the particular example we are considering.

$$\begin{aligned}\chi < \ & \left\|\sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b}\left(\mathbf{r}_{\tau,i}^*\frac{\partial \mathbf{r}_{\tau,i}^T}{\partial\tau} + \frac{\partial \mathbf{r}_{\tau,i}^*}{\partial\tau}\mathbf{r}_{\tau,i}^T - \frac{\partial R_{\tau,i}}{\partial\tau}\right)\right\| \|\mathbf{f}+\mathbf{f}^*\| \\ & + \left\|\sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b}\left(\frac{\partial \mathbf{P}_{\tau,i}}{\partial\tau} - a_{i-\delta}\frac{\partial \mathbf{r}_{\tau,i}^*}{\partial\tau}\right)\right\|\end{aligned}$$

$$\begin{aligned}\chi < \ & \left\|\sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b}\left(\mathbf{r}_{\tau,i}^*\frac{\partial \mathbf{r}_{\tau,i}^T}{\partial\tau} + \frac{\partial \mathbf{r}_{\tau,i}^*}{\partial\tau}\mathbf{r}_{\tau,i}^T - \frac{\partial R_{\tau,i}}{\partial\tau}\right)\right\| \|\mathbf{f}+\mathbf{f}^*\| \\ & + \left\|\sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b}\left(\frac{\partial \mathbf{P}_{\tau,i}}{\partial\tau} - a_{i-\delta}\frac{\partial \mathbf{r}_{\tau,i}^*}{\partial\tau}\right)\right\|\end{aligned}$$

We separate this into two terms, one which remains even when the equalizer is at its optimal location, and one which is proportional to the error in the Equalizer, which we have bounded by r_b.

$$\begin{aligned}\chi &< \Psi(k,\tau)\|\mathbf{f}+\mathbf{f}^*\| + \Phi(k,\tau) \\ &< \Psi(k,\tau)(\|\mathbf{f}_k^*\| + \|\mathbf{f}_k\|) + \Phi(k,\tau) \\ &< \Psi(k,\tau)(\|\mathbf{f}_k^*\| + r_b) + \Phi(k,\tau)\end{aligned}$$

where

$$\Psi(k,\tau) = \left\|\sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b}\left(\mathbf{r}_{\tau,i}^*\frac{\partial \mathbf{r}_{\tau,i}^T}{\partial\tau} + \frac{\partial \mathbf{r}_{\tau,i}^*}{\partial\tau}\mathbf{r}_{\tau,i}^T - \frac{\partial R_{\tau,i}}{\partial\tau}\right)\right\|$$

$$\Phi(k,\tau) = \left\|\sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b}\left(\frac{\partial \mathbf{P}_{\tau,i}}{\partial\tau} - a_{i-\delta}\frac{\partial \mathbf{r}_{\tau,i}^*}{\partial\tau}\right)\right\|$$

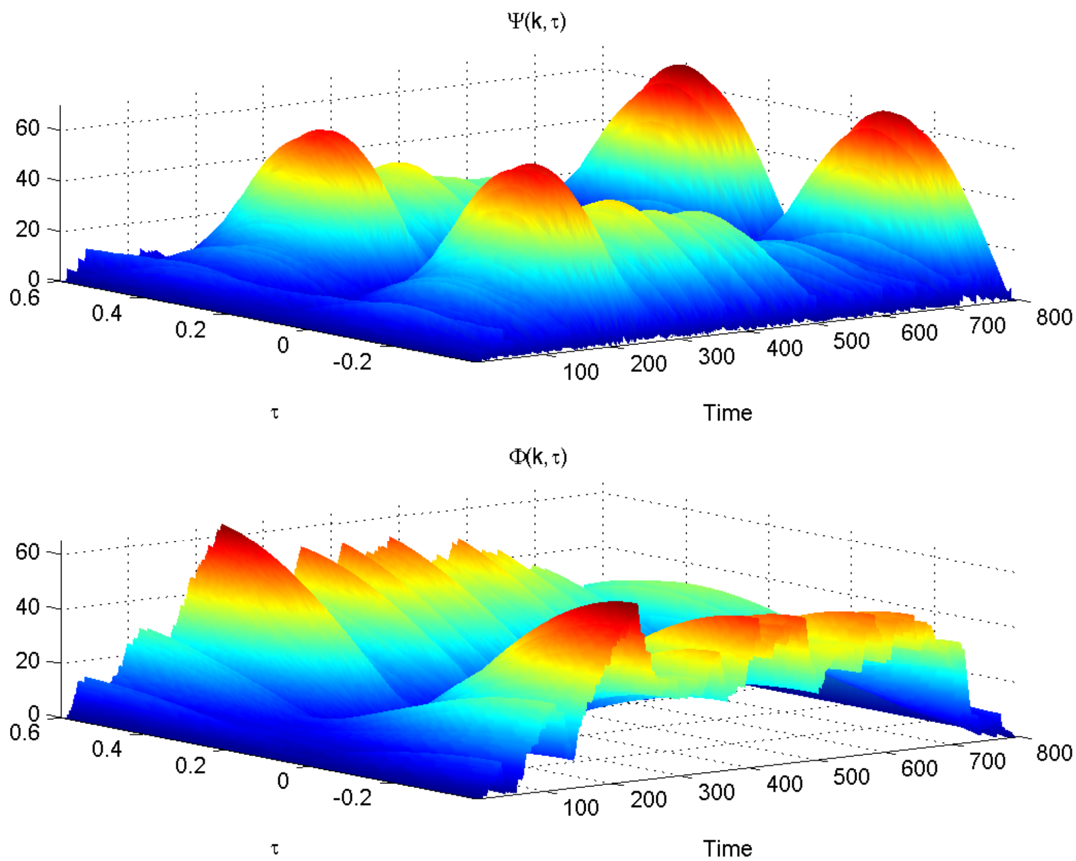

Figure 5.6: The two functions determining the Lipschitz Constant that gives the sensitivity of the total perturbation to the timing recovery.

These two functions are shown in Figure 5.6. Note that the size of the total perturbation depends on the length of time we are considering, since the function is growing with time.

- **Boundedness of total perturbation for the equalizer**

$$\|p_n(k,0,0)\| \leq B_{p,b}$$

$$\|p_n(k,0,0)\| \leq \left\| \sum_{i=nT_b/\mu_b}^{k+nT_b/\mu_b} \left[\left(a_{i-\delta} - \mathbf{r}_{\tau,i}^T \mathbf{f}_i^* \right) \mathbf{r}_{\tau,i}^* \right] \right\|$$

We plot this for the example we have chosen in Figure 5.7. We can see that we could conservatively bound $p_n(k,0,0)$ with $B_{p,b} = .01$. Note that,

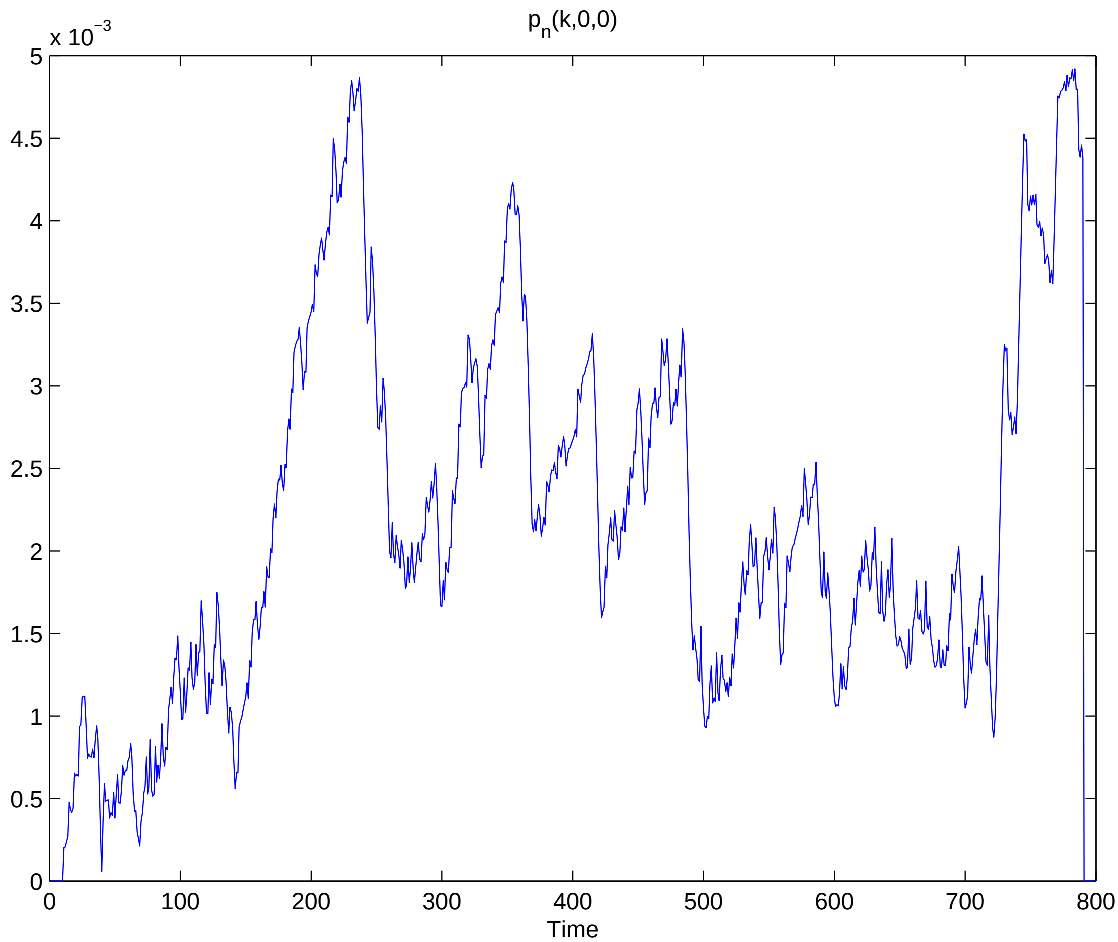

Figure 5.7: The Total Perturbation in the Equalizer Over Time

numerically, this particular bound is very sensitive to errors. That is to say, if the accuracy in our calculation of $\mathbf{f}^*$ and τ_k^* is not very good, then there will still be some mean gradient attempting to push us to the exact $\mathbf{f}_k^*$, and this will show up as growth of $\|p_n(k,0,0)\|$ over time. Within the accuracy of the simulations used $(0 \approx 10^-7)$, Figure 5.7 shows that over a reasonable time frame the total perturbation does not appear to grow.

- **Continuity of Averaged Equalizer to changes in timing**

$$\|\bar{b}_k + \mu_b f_{av,b}(\bar{b}_k, \bar{a}_k) - \left(\bar{b}_k + \mu_b f_{av,b}(\bar{b}_k, \bar{a}'_k)\right)\|$$
$$= |\mu_b|\|f_{av,b}(\bar{b}_k, \bar{a}_k) - f_{av,b}(\bar{b}_k, \bar{a}'_k)\| \leq \mu_b \gamma \|\bar{a}_k - \bar{a}'_k\|$$

We had that the averaged error system for the equalizer was given by

$$\bar{\mathbf{f}}_{k+1} = \bar{\mathbf{f}}_k + \mu_b \left(\mathbf{P}_{\tau_k,k} - R_{\tau_k,k}\left(\bar{\mathbf{f}}_k + \mathbf{f}^*_k\right)\right) + \mathbf{f}^*_k - \mathbf{f}^*_{k+1}$$
$$\gamma \leq \left\| \frac{\partial \mathbf{P}_{\tau,k}}{\partial \tau} - \frac{\partial R_{\tau,k}}{\partial \tau}\left(\bar{\mathbf{f}}_k + \mathbf{f}^*_k\right)\right\|$$

Thus, we can use (5.24) and (5.25) to find this Lipschitz constant as well.

Now that we have calculated the parameters, we wish to see if the SFFBAC Hovering Theorem applies to this situation. Our hovering theorem tells us that, since we have no time variation in the desired trajectories in this case, as we shrink the step size to zero, the true system trajectories should approach the desired (averaged system) trajectories. Figure 5.8 compares the averaged and unaveraged trajectories for several pairs of μ. The fact that the smaller pairs of μ values have error closer to zero verifies that the qualitative information provided by the theorem is true. Figure 5.9 compares the bound provided by the theorem with the actual error between the actual and desired trajectories.

5.2.3 Misbehavior

Finally, our misbehavior theorem predicts that, if we make our timing algorithm move so slowly that it can not track a desired periodic trajectory, we have the possibility that the equalizer parameters will move periodically in a manifold away from the point that was chosen as to have "good" performance. We wish to investigate what this means in the case of a particular timing-equalizer pair. Instead of

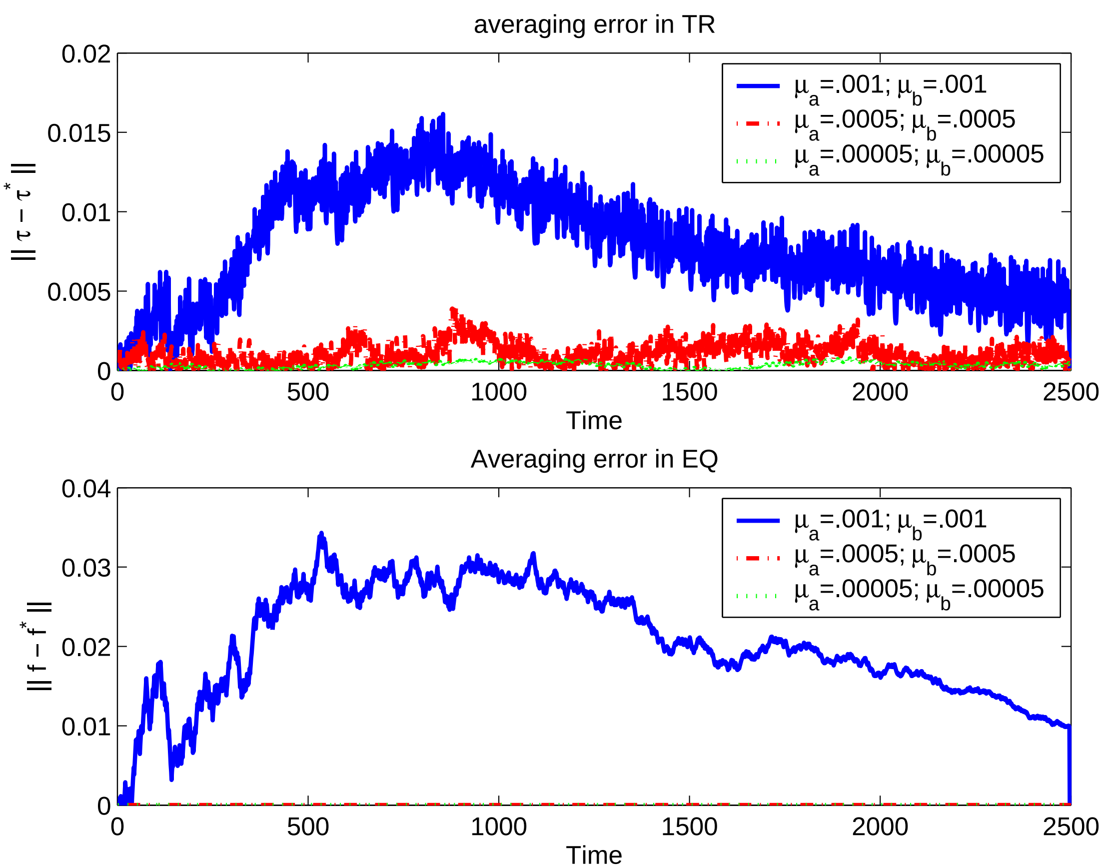

Figure 5.8: A trajectory locking SFFBAC example from digital receivers.

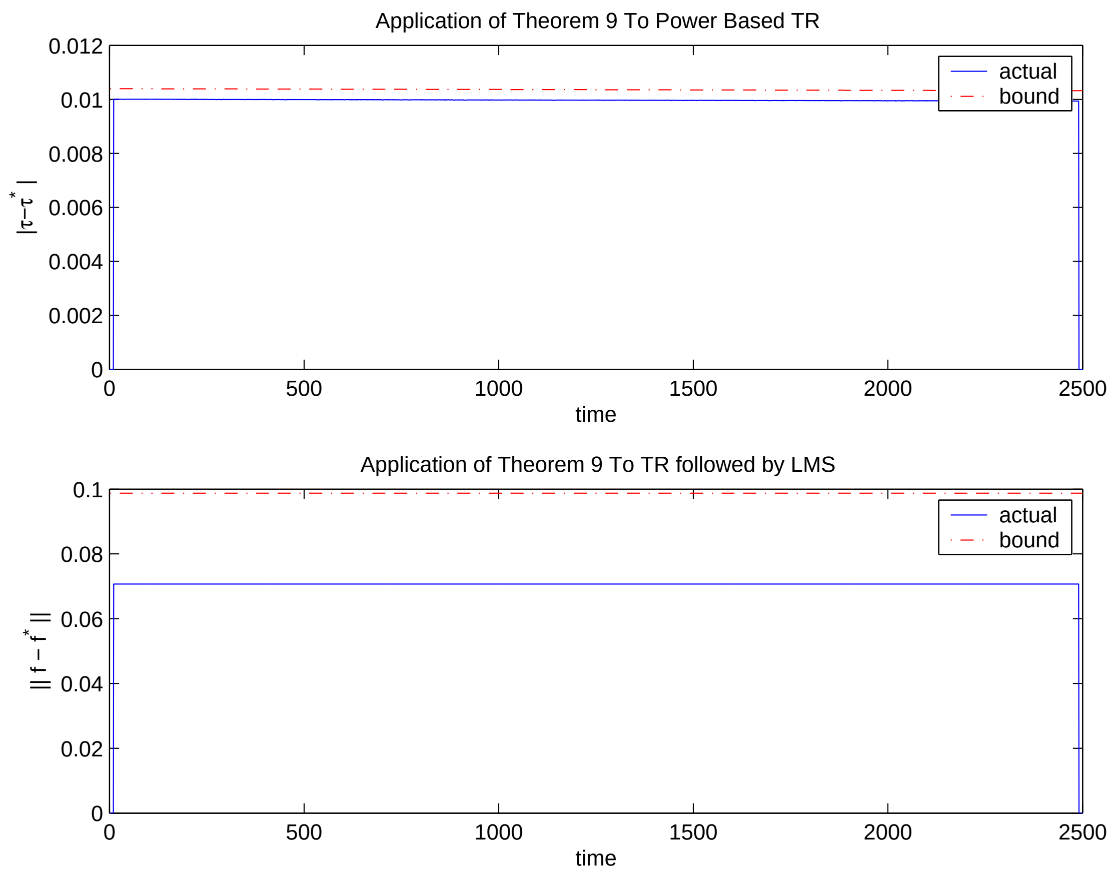

Figure 5.9: Bounds compared with actual parameter descents for $\mu_a = 10^-3$ and $\mu_b = 10^-4$.

calculating the relevant constants, this time, we merely verify their existence, and then use μ shrinking arguments. Since the bounds provided by even the averaging theorems are typically not very tight, this is a more typical use of this type of theorem. Once again we consider a power based timing recovery followed by a LMS equalizer. All of our Lipschitz and boundedness properties are guaranteed by the fact that we have chosen two continuously differentiable adaptive state functions with respect to both the equalizer and the timing parameters[4]. The remaining difficulty is only to ensure that the averaged adaptive element state functions are contractive to zero error. We have already established this fact for the Power-based timing recovery followed by a LMS equalizer for a particular timing offset and channel combination in Section 5.2, and we use the same set up here, only now with a slow variation around the fixed timing offset we previously had. We then verify for this example via simulation that we see exactly what Theorem 10 predicts. Looking at Figure 5.10 we see just what our theorem predicts for a timing step size which is too small. The timing recovery does not track the sinusoidal timing offset quickly enough, and settles in near a constant timing parameter. The equalizer then moves in a zero manifold, fluctuating around its desired values (which are the means which we have subtracted).

5.3 Conclusions

We see that, while we want smaller step sizes to decrease averaging error in adaptive devices, we pay a price for a smaller step size of decreased ability to track time variation. The misbehavior shown in Figure 5.10 is a perfect example of this

[4]Differentiable $\Rightarrow$ Lipschitz continuous on a compact set (ball) and continuous $\Rightarrow$ bounded on a compact set.

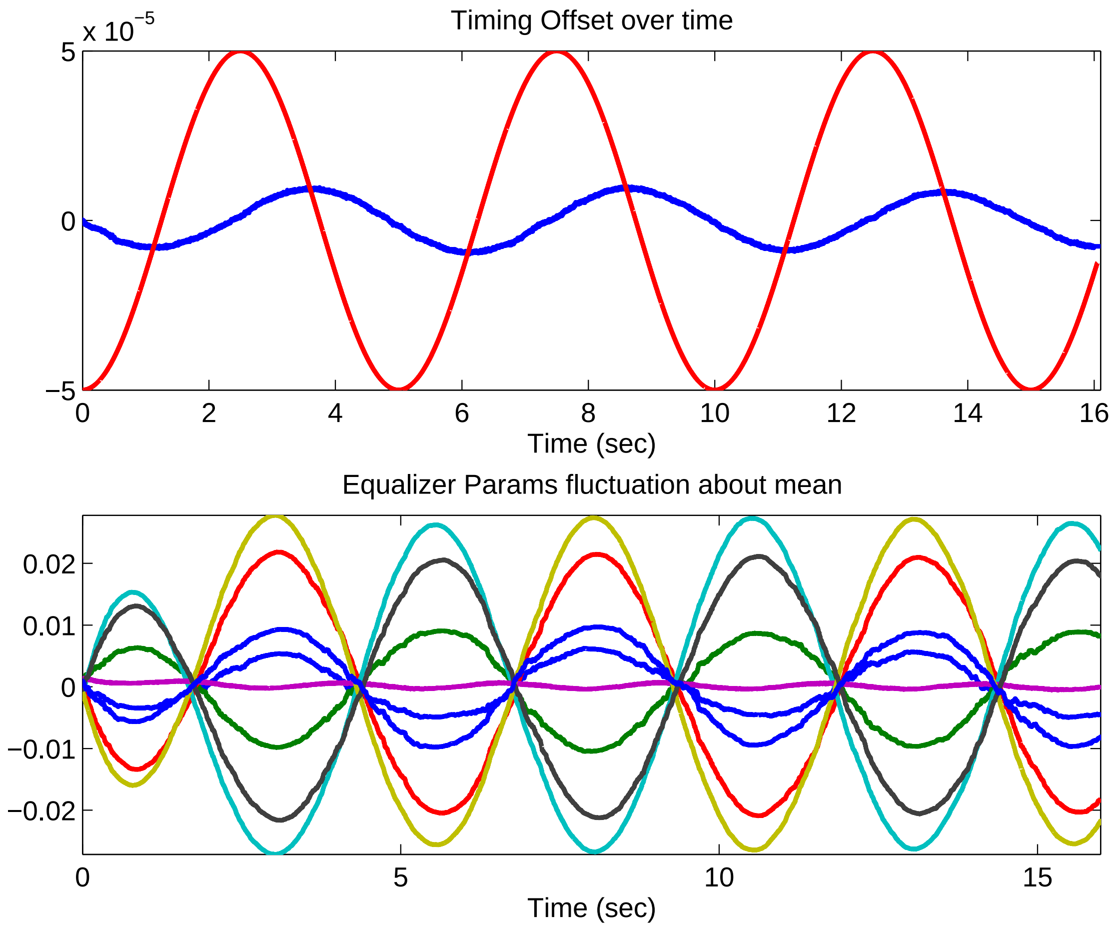

Figure 5.10: Misbehavior in a Power based timing recovery scheme followed by LMS equalization.

phenomenon, where almost all of the averaging error is gone, but both the timing recovery and the equalization do not move to their desired points, since the timing recovery can not track the time variation in the timing offset fast enough. The important information that our analysis has given us is that the inability of the first element to track its desired point can cause the second element to misbehave. The tendency for such a phenomenon to occur has been linked, [22], to the size of sensitivity (χ or $\mu_b\gamma$) through Theorems 7, 8, and 9. And from this, we can deduce that algorithms with lower sensitivities (but the same stationary points) will be better second elements in SFFBACs. We investigated an example of this phenomenon in section 5.1 by comparing two schemes for carrier recovery in a digital receiver that contained an automatic gain control followed by an carrier phase recovery.

Chapter 6

Results Summary and Conclusions

We began this thesis by introducing the notion of an adaptive element and providing an arsenal of theorems to characterize the way adaptive elements behave from a deterministic stability theory standpoint. Examples of adaptive devices found in digital receivers were then analyzed, removing any assumptions about idealities in other adaptive processing in the receiver. We saw that in general, the behavior of adaptive state equations for adaptive signal processing elements commonly found in adaptive receivers depends on the other adaptive devices that are included in the receiver. Then, referring to an appendix containing quotes about the interaction of these devices which suggested a lack of a general theory, we found practical evidence that engineers encounter interactions among adaptive elements in the systems they build.

We were then inspired to investigate the way that multiple adaptive signal processing elements behave when they are connected together. To begin an attack of this large problem, we selected a particular binary adaptive compound structure: the series feedforward binary adaptive compound (SFFBAC). To characterize the behavior of SFFBACs, we provided a set of qualitative guidelines that guarantee proper operation. We then showed that these qualitative guidelines could be connected to rigorous mathematical quantitative guidelines, and derived a version of all of the theorems in the single algorithm case for the SFFBAC case. Along the way, we gained insight into the way the two adaptive elements in a SFFBAC interact.

From our theoretical work on SFFBACs, we learned that both jitter from disturbances as well as averaging error from the earlier element in a SFFBAC can couple into the second element's state and produce jitter, through a sensitivity constant χ (aka $\mu_b \gamma$). This suggests that second elements which are less sensitive to the first element's parameters with equally good stationary points were better than those that are more sensitive. From an averaging theory standpoint, the coupling of averaging error in the earlier adaptive element into later adaptive elements indicated that the step sizes chosen in the first element in a SFFBAC affect the jitter in the second element, which argues for decreasing (effective) step sizes as you move further along down a chain of serially connected adaptive elements.

We also provided a characterization of possible misbehavior in SFFBACs, where the adaptation (or lack of adaptation) of the first element caused the second element to misbehave. After we proved this theorem, we showed an example in which two digital receiver components, a timing recovery unit and an equalizer, underwent the misbehavior described in the theorem.

Now, we sit at the border between past and future work.

Chapter 7

Future Work

The results in this thesis can be extended in a multitude of ways. First of all, one could investigate more fully specific systems which employ distributed adaptation. We could continue on characterizing the behavior of particular types of signal processing pairs. Possible pairs from digital receivers to investigate include: automatic gain control and phase correction, auto-regressive whitening followed by CMA equalization, adaptive beam-forming followed by adaptive equalization, beam-forming and power control, gain control and echo cancellation, etc. Each of these applications of the theory could serve as motivating examples with which to extend the basic results provided here to more specific situations. While such extensions would inevitably narrow the applicability of the results, they could also serve to broaden the implications of the theory.

More fundamentally, the characterization here of the behavior of series feed-forward binary adaptive compounds is far from complete. The theorems provided here all gave sufficient conditions for good behavior. Thus, an interesting topic would be to explore conditions which are also necessary for good behavior. When addressing questions of necessity from a nonlinear stability theory standpoint, one inevitably encounters Lyapunov techniques [8], [6], [7]. Thus, an immediate, and important, enlargement of this theory is to develop and include the relevant theorems for using Lyapunov techniques to describe the stability of adaptive compounds. Since there are a wealth of Lyapunov based results for single adaptive elements, the majority of the relevant work would be to extend the ideas to consider the coupling of two adaptive systems. Here, a large body of dynamical systems literature could come in handy.

Yet, the study of series feed-forward binary adaptive compounds need not be done entirely from a deterministic stability theory mindset. Continuing from a dynamical systems mindset the first natural (and probably necessary) alternative method of viewing this problem involves using stochastic tools. Many of the theorems that have described stochastic adaptation for just a single algorithm are developed with techniques similar to the proofs used in the deterministic theory, characterizing stochastic adaptation by describing the evolution of means and variances. Yet, when we add stochastic concepts, we have not only moments, but whole densities, which often can not be found in closed form, to deal with. The rigorous theory of the behavior of stochastic difference equations is still somewhat in its infancy, and thus, along this route it would be possible to conduct some difficult, yet broadly impacting research.

Yet, one need not study the behavior of adaptive compounds through the dynamical systems magnifying glass alone. There are a number of other fields of research which can be directly applied to this problem. If we look at the distributed adaptation as competing optimizations, immediately a framework for game-theoretic discussion pops out. Thus, it could be fascinating to investigate applying game-theoretic ideas to binary adaptive compounds.

Of course, whatever one does for specific binary adaptive structures, SFFBACs for instance, one could investigate for other structures. Once one is considering this problem along the structural level, the possibilities become endless. A researcher who truly understands the way several adaptive elements will behave when hooked together ought to have great insight into the study of complex adaptive systems. Once one can do adequate research in this realm, one could begin to investigate anything from the economy to the human visual system. The possibilities are endless.

Appendix A

Sensitivity Analysis for Digital PAM/QAM Receivers

In this appendix we provide the derivations for the sensitivity functions listed in Tables 2.1-2.4. For sake of brevity, we define the input to the following algorithms to be of the form

$$x_k = g_k e^{j\hat{\theta}_k} \sum_n a_n h_{\hat{\tau}_k,\mathbf{f}_k}[k,n] + v_k \tag{A.1}$$

where the impulse response is given by

$$h_{\hat{\tau},\mathbf{f}} = \sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-l)T + \hat{\tau}_k) p((k-l-n)T + \hat{\tau}_k - \tau_i) f_l \tag{A.2}$$

We will remove the adaptive state of the device (e.g. timing recovery, equalization, etc.) which we are studying from this equation when we calculate a particular algorithm's sensitivity. Furthermore, if the system of interest does not use one of the devices we've included to process the input to the device of interest, we can simply remove that devices parameter from the input equation.

A.1 Gain Control

We are now considering adaptive algorithms that adjust the gain. Using A.1, the general input to a gain control can be written as

$$x_k = e^{j\hat{\theta}_k} \sum_n a_n h_{\hat{\tau}_k,\mathbf{f}_k}[k,n] + v_k \tag{A.3}$$

where $h_{\hat{\tau}_k,\mathbf{f}_k}[k,n]$ is defined as in (A.2). Using this input, we now develop the sensitivity function corresponding to (2.29)

$$S_g(\hat{\theta}, \hat{\tau}, \mathbf{f}, g) = \mathsf{E}\left[\left(g_k^2|x_k|^2 - d\right) sign[g_k]\right] \tag{A.4}$$

One of the key terms in this expectation is

$$\mathsf{E}\left[|x_k|^2\right] = \mathsf{E}\left[\left(\sum_n a_n e^{-j\hat{\theta}_k} h_{\hat{\tau},\mathbf{f}}[k,n] + e^{-j\hat{\theta}} v_k\right)\left(\sum_{n_2} a_{n_2}^* e^{j\hat{\theta}_k} h_{\hat{\tau},\mathbf{f}}^*[k,n_2] + e^{j\hat{\theta}_k} v_k^*\right)\right]$$

which, under standard assumptions of iid zero mean source symbols that are uncorrelated with the iid noise, becomes

$$\mathsf{E}\left[|x_k|^2\right] = \mathsf{E}[|a|^2] \sum_n |h_{\hat{\tau},\mathbf{f}}[k,n]|^2 + \mathsf{E}[|v|^2]$$

Substituting this into (A.4), we have

$$S_g(\hat{\theta}, \hat{\tau}, \mathbf{f}, g) = \left(g_k^2 \left(\mathsf{E}[|a|^2] \sum_n |h_{\hat{\tau},\mathbf{f}}[k,n]|^2 + \mathsf{E}[|v|^2]\right) - d\right) sign[g_k]$$

which is the sensitivity function for the automatic gain control that we will be considering.

A.2 Carrier Phase Tracking

Next, we turn to calculating sensitivity functions for some of the algorithms listed in Table 2.1 that adjust the carrier phase. A general (possibly processed by all of the other adaptive elements we are considering) input to a carrier phase recovery unit is

$$x_k = g_k \sum_n a_n h_{\hat{\tau}_k,\mathbf{f}_k}[k,n] + v_k$$

where $h_{\hat{\tau}_k,\mathbf{f}_k}[k,n]$ is defined as in (A.2). Now that we have defined the input, we begin by developing the sensitivity of the trained QAM Costas loop

$$S_\theta(\hat{\theta}, \hat{\tau}, \mathbf{f}, g) = \mathsf{E}\left[\mathrm{Im}\left\{a_k^* e^{-j\hat{\theta}} x_k\right\}\right]$$

$$S_\theta(\hat{\theta},\hat{\tau},\mathbf{f},g) = \mathsf{E}\left[\text{Im}\left\{a_k^* e^{-j\hat{\theta}}\left(g_k \sum_n a_n h_{\hat{\tau},\mathbf{f}}[k,n] + g_k \sum_{l=0}^{P-1} v_{k-l} f_l\right)\right\}\right]$$

Using the property of complex numbers, $\text{Im}\{\alpha\} = \frac{1}{2}\left[\alpha - \alpha^*\right]$, gives

$$S_\theta(\hat{\theta},\hat{\tau},\mathbf{f},g) = \begin{array}{l} \frac{1}{2}\mathsf{E}\left[a_k^* e^{-j\hat{\theta}}\left(g_k \sum_n a_n h_{\hat{\tau},\mathbf{f}}[k,n] + g_k \sum_{l=0}^{P-1} v_{k-l} f_l\right)\right. \\ \left. -a_k e^{j\hat{\theta}}\left(g_k \sum_n a_n^* h_{\hat{\tau},\mathbf{f}}^*[k,n] + g_k \sum_{l=0}^{P-1} v_{k-l}^* f_l^*\right)\right] \end{array}$$

Using standard assumptions of a zero mean iid source uncorrelated from zero mean iid noise gives

$$S_\theta(\hat{\theta},\hat{\tau},\mathbf{f},g) = g_k \mathsf{E}[|a|^2]\text{Im}\left\{e^{-\hat{\theta}} h_{\hat{\tau},\mathbf{f}}^*[k,k]\right\}$$

writing the imaginary part operation another way gives

$$S_\theta(\hat{\theta},\hat{\tau},\mathbf{f},g) = g_k \mathsf{E}[|a|^2]\left|h_{\hat{\tau},\mathbf{f}}[k,k]\right| \sin\left(\angle h_{\hat{\tau},\mathbf{f}}[k,k] - \hat{\theta}\right)$$

Which is the sensitivity function for the trained Costas Loop, which is the first entry in Table 2.1.

Next we attempt to derive the sensitivity function for the last entry in the Table, which works well for QPSK and okay for QAM16

$$S_\theta(\hat{\theta},\hat{g},\mathbf{f},\hat{\tau}) = \mathsf{E}\left[e(k)|\hat{\theta},\hat{g},\mathbf{f},\hat{\tau}\right] = \mathsf{E}\left[\text{Im}\left\{x_k^4 e^{-j4\hat{\theta}}\right\}\right] \tag{A.5}$$

To simplify the ensuing discussion, we first determine one of the important terms

$$\begin{aligned} \mathsf{E}\left[x_k^4\right] = \;& \mathsf{E}\left[\hat{g}_k^4\left(\sum_n \sum_l \sum_m \sum_p a_{k-n}a_{k-l}a_{k-m}a_{k-p}h_{\hat{\tau},\mathbf{f}}^4[k]\right.\right. \\ & +4v_k \sum_l \sum_m \sum_p a_{k-l}a_{k-m}a_{k-p}h_{\hat{\tau},\mathbf{f}}^3[k] \\ & \left.\left.+4v_k^2 \sum_m \sum_p a_{k-m}a_{k-p}h_{\hat{\tau},\mathbf{f}}^2[k] + 4v_k^3 \sum_p a_{k-p}h_{\hat{\tau},\mathbf{f}}[k] + v_k^4\right)\right] \end{aligned} \tag{A.6}$$

$$\begin{aligned} \mathsf{E}\left[x_k^4\right] = \;& \hat{g}_k^4\left(\sum_n \sum_l \sum_m \sum_p \mathsf{E}\left[a_{k-n}a_{k-l}a_{k-m}a_{k-p}\right] h_{\hat{\tau},\mathbf{f}}^4[k]\right. \\ & \left.+4\mathsf{E}\left[v_k^2\right] \sum_m \sum_p \mathsf{E}\left[a_{k-m}a_{k-p}\right] h_{\hat{\tau},\mathbf{f}}^2[k] + \mathsf{E}\left[v_k^4\right]\right) \end{aligned} \tag{A.7}$$

$$\begin{aligned} \mathsf{E}\left[x_k^4\right] = \;& \sum_n \mathsf{E}\left[a_{k-n}^4\right] h_{\hat{\tau},\mathbf{f}}^4[k] + 3\sum_l \sum_{m\neq l} \mathsf{E}[a_l^2]\mathsf{E}[a_m^2] h_{\hat{\tau},\mathbf{f}}^4[k] + \\ & 4\mathsf{E}\left[v_k^2\right] \sum_m \mathsf{E}[a_{k-m}] h_{\hat{\tau},\mathbf{f}}^2[k] + \mathsf{E}\left[v_k^4\right] \end{aligned} \tag{A.8}$$

Assuming that the noise, v, is circularly Gaussian makes $\mathsf{E}[v_k^2] = 0$ and $\mathsf{E}[v_k^4] = 0$, so we have

$$\mathsf{E}\left[x_k^4\right] = \sum_n \mathsf{E}\left[a_{k-n}^4\right] h_{\hat{\tau},\mathbf{f}}^4[k] + 3\sum_l \sum_{m\neq l} \mathsf{E}[a_l^2]\mathsf{E}[a_m^2] h_{\hat{\tau},\mathbf{f}}^4[k] \tag{A.9}$$

Also, if we assume the source constellation is such that

- the variance of the real part is equal to the variance of the imaginary part (symmetric constellations)
- the real part of the symbols are chosen independently of the imaginary part
- the real part and the imaginary part are zero mean.

as is the case with many constellations (square QAM, etc.), then, we have $\mathsf{E}[a_k^2] = 0$, and

$$\mathsf{E}\left[x_k^4\right] = \sum_n \mathsf{E}\left[a_{k-n}^4\right] h_{\hat{\tau},\mathbf{f}}^4[k] \tag{A.10}$$

Now we use this fact and the assumptions to get the sensitivity function

$$\begin{aligned} S_\theta(\hat{\theta}, \hat{g}, \mathbf{f}, \hat{\tau}) &= \tfrac{1}{2}\left(\mathsf{E}\left[x_k^4 e^{-j4\hat{\theta}}\right] + \mathsf{E}\left[(x_k^4)^* e^{j4\hat{\theta}}\right]\right) \\ &= \left|\mathsf{E}\left[a^4\right] \textstyle\sum_n h_{\hat{\tau},\mathbf{f}}^4[k,n]\right| \sin\left(\angle \textstyle\sum_n h_{\hat{\tau},\mathbf{f}}^4[k,n] + \angle\mathsf{E}\left[a^4\right] - \hat{\theta}_k\right) \end{aligned} \tag{A.11}$$

A.3 Timing Recovery

We will begin by calculating some sensitivity functions for the update terms in Table 2.2. Many of these update terms use the signal

$$y(kT+\hat{\tau}) = \hat{g}_k e^{-j\hat{\theta}_k} \sum_n a_n \sum_{l=0}^{P-1} \sum_{i=1}^{Q} c_i((k-l)T+\hat{\tau}_k) p((k-l-n)T+\hat{\tau}_k-\tau_i) f_l + v_k \tag{A.12}$$

Which has been written to include the effects of the other parameters (under the same assumptions on slow movement as for carrier recovery) on the timing update function.

We begin by deriving the Mueller-Mueller sensitivity function

$$S_\tau(\hat{\tau},\hat{\theta},\mathbf{f},\hat{g}) = \mathsf{E}\left[\mathrm{Re}\left\{\hat{a}^*_{k-1}y(kT+\hat{\tau}_k) - \hat{a}^*_k y((k-1)T+\hat{\tau}_k)\right\}|\hat{\tau},\hat{\theta},\mathbf{f},\hat{g}\right] \quad \text{(A.13)}$$

$$\mathsf{E}\left[\hat{a}^*_{k-1}y(kT+\hat{\tau}_k)\right] = \hat{g}_k e^{-j\hat{\theta}_k}\mathsf{E}[|a|^2]\sum_{l=0}^{P-1}\sum_{i=1}^{Q} c_i((k-l)T+\hat{\tau}_k)p((1-l)T+\hat{\tau}_k-\tau_i)f_{l,k} \quad \text{(A.14)}$$

$$\begin{aligned}\mathsf{E}\left[\hat{a}^*_k y((k-1)T+\hat{\tau}_{k-1})\right] = \; & \hat{g}_{k-1}e^{-j\hat{\theta}_{k-1}}\mathsf{E}[|a|^2]\textstyle\sum_{l=0}^{P-1}\sum_{i=1}^{Q} \\ & c_i((k-1-l)T+\hat{\tau}_{k-1})p((-1-l)T+\hat{\tau}_{k-1}-\tau_i)f_{l,k-1}\end{aligned} \quad \text{(A.15)}$$

$$\begin{aligned}S_\tau(\hat{\tau},\hat{\theta},\mathbf{f},\hat{g}) = \; & \mathrm{Re}\Big\{\hat{g}_k e^{-j\hat{\theta}_k}\mathsf{E}[|a|^2]\textstyle\sum_{l=0}^{P-1}\sum_{i=1}^{Q} \\ & c_i((k-l)T+\hat{\tau}_k)p((1-l)T+\hat{\tau}_k-\tau_i)f_{l,k} \\ & -\hat{g}_{k-1}e^{-j\hat{\theta}_{k-1}}\mathsf{E}[|a|^2]\textstyle\sum_{l=0}^{P-1}\sum_{i=1}^{Q} \\ & c_i((k-1-l)T+\hat{\tau}_{k-1})p((-1-l)T+\hat{\tau}_{k-1}-\tau_i)f_{l,k-1}\}\end{aligned} \quad \text{(A.16)}$$

Next we derive the sensitivity function for the zero crossing timing error detector

$$S_\tau(\hat{\tau},\hat{\theta},\mathbf{f},\hat{g}) = \mathsf{E}\left[\mathrm{Re}\left\{\left(\hat{a}^*_{k-1}-\hat{a}^*_k\right)y(kT-T/2+\hat{\tau}_{k-1})e^{-j\hat{\theta}_k}\right\}|\hat{\tau},\hat{\theta},\mathbf{f},\hat{g}\right] \quad \text{(A.17)}$$

$$\begin{aligned}S_\tau(\hat{\tau},\hat{\theta},\mathbf{f},\hat{g}) = \; & \mathrm{Re}\Big\{\hat{g}_k e^{-j\hat{\theta}_k}\mathsf{E}\left[|a|^2\right]\textstyle\sum_{l=0}^{P-1}\sum_{i=1}^{Q} c_i((k-l-\frac{1}{2})T+\hat{\tau}_{k-1}) \\ & \left[p((\tfrac{1}{2}-l)T+\hat{\tau}_{k-1}-\tau_i)-p((-\tfrac{1}{2}-l)T+\hat{\tau}_{k-1}-\tau_i)\right]f_{l,k}\Big\}\end{aligned} \quad \text{(A.18)}$$

Next we compute the sensitivity function for the early late detector.

$$S_\tau(\hat{\tau}, \hat{\theta}, \mathbf{f}, \hat{g}) = \mathsf{E}\left[\mathrm{Re}\left\{\hat{a}_k^*\left(y((k+\frac{1}{2})T + \hat{\tau}_k) - y((k-\frac{1}{2})T + \hat{\tau}_{k-1})\right)\right\}\right] \quad \text{(A.19)}$$

$$\begin{aligned} S_\tau(\hat{\tau}, \hat{\theta}, \mathbf{f}, \hat{g}) = \; & \mathrm{Re}\left\{\hat{g}_k e^{-j\hat{\theta}_k}\mathsf{E}[|a|^2]\left(\textstyle\sum_{l=0}^{P-1}\sum_{i=1}^{Q} c_i((k-l+\frac{1}{2})T + \hat{\tau}_k)\right.\right. \\ & p((\tfrac{1}{2}-l)T + \hat{\tau}_k - \tau_i)f_{l,k} - \textstyle\sum_{l=0}^{P-1}\sum_{i=1}^{Q} c_i((k-l-\frac{1}{2})T + \hat{\tau}_{k-1}) \\ & \left.\left. p((-\tfrac{1}{2}-l)T + \hat{\tau}_{k-1} - \tau_i)f_{l,k}\right)\right\} \end{aligned} \quad \text{(A.20)}$$

Next we compute the sensitivity function for the special case of M-nonlinearity timing recovery for M=2.

$$\tau_{k+1} = \tau_k + \mu \mathrm{Re}\left\{y_{\hat{\tau}_k} y'_{\hat{\tau}_k}\right\} \quad \text{(A.21)}$$

$$y_{\hat{\tau}_k} = g_k e^{-j\hat{\theta}_k}\sum_n a_n h_{\tau,\mathbf{f}}[k,n] + v_k \quad \text{(A.22)}$$

$$y'_{\hat{\tau}_k} = g_k e^{-j\hat{\theta}_k}\sum_n a_n dh_{\tau,\mathbf{f}}[k,n] + v_k \quad \text{(A.23)}$$

$$\mathsf{E}\left[y^*_{\hat{\tau}_k} y'_{\hat{\tau}_k}\right] = |g_k|^2 \mathsf{E}[|a_n|^2]\sum_n h^*_{\tau,\mathbf{f}}[k,n]\, dh_{\tau,\mathbf{f}}[k,n] + \sigma^2 \quad \text{(A.24)}$$

$$S_\tau(\hat{\theta}, \mathbf{f}, g, \hat{\tau}) = |g_k|^2 \mathsf{E}[|a_n|^2]\sum_n \mathrm{Re}\left\{h^*_{\tau,\mathbf{f}}[k,n]\, dh_{\tau,\mathbf{f}}[k,n]\right\} + \sigma^2 \quad \text{(A.25)}$$

$$h_{\hat{\tau},\mathbf{f}}[k,n] = \sum_{l=0}^{P-1}\sum_{i=1}^{Q} c_i((k-l)T + \hat{\tau}_k)p((k-l-n)T + \hat{\tau}_k - \tau_i)f_l \quad \text{(A.26)}$$

$$dh_{\hat{\tau},\mathbf{f}}[k,n] = \sum_{l=0}^{P-1}\sum_{i=1}^{Q}\left[dc_i((k-l)T+\hat{\tau}_k)p((k-l-n)T+\hat{\tau}_k-\tau_i) + c_i((k-l)T+\hat{\tau}_k)dp((k-l-n)T+\hat{\tau}_k-\tau_i)\right]f_l \quad \text{(A.27)}$$

A.4 Equalization

We now study the sensitivities of equalization algorithms to other devices parameters in a digital receiver. Handling the possibility that all of the other devices have processed the input to the equalizer, we have a general input of the form

$$x_k = \hat{g}_k e^{-j\hat{\theta}_k}\sum_n a_n h_{\hat{\tau}}[k,n] + v_k \quad \text{(A.28)}$$

Define $\mathbf{f} = [f_0 f_1 \cdots f_{P-1}]^T$ and $\mathbf{r}_k = [x_k x_{k-1} \cdots x_{k-P+1}]^T$. First we analyze the LMS update sensitivity function

$$S_{f_m}(\hat{g},\hat{\theta},\hat{\tau},\mathbf{f}) = \mathsf{E}\left[x_{k-m}\left(a_{k-\delta} - \sum_{l=0}^{P-1} x_{k-l}f_l\right)\right] \quad \text{(A.29)}$$

$$S_{f_m}(\hat{g},\hat{\theta},\hat{\tau},\mathbf{f}) = \mathsf{E}\left[x^*_{k-m}a_{k-\delta}\right] - \sum_{l=0}^{P-1}\mathsf{E}\left[x_{k-l}x^*_{k-m}\right]f_l \quad \text{(A.30)}$$

using standard iid zero mean assumptions about the source we have

$$S_{f_m}(\hat{g},\hat{\theta},\hat{\tau},\mathbf{f}) = \hat{g}_{k-m}e^{j\hat{\theta}_{k-m}}\mathsf{E}[|a|^2]h^*_{\hat{\tau}}[k-m,k-\delta] - \sum_{l=0}^{P-1}\hat{g}_{k-m}\hat{g}_{k-l}e^{j(\hat{\theta}_{k-m}-\hat{\theta}_{k-l})} \sum_n \mathsf{E}[|a|^2]h^*_{\hat{\tau}}[k-m,n]h_{\hat{\tau}}[k-l,n]f_l + \mathsf{E}[v^*_{k-m}v_{k-l}]f_l \quad \text{(A.31)}$$

Next we derive the sensitivity equation for (2,2) CMA. We begin our analysis with

$$S_{f_m}(\mathbf{f},\tau,\theta,g) = \mathsf{E}\left[(\gamma - |\textstyle\sum_i f_i\left(\sum_n a_n h[k-i,n]+v_{k-i}\right)|^2) \textstyle\sum_i f_i\left(\sum_n a_n h[k-i,n]+v_k\right)\left(\sum_n a_n h[k-m,n]+v_k\right)\right]$$

So our analysis breaks into finding two terms

$$\mathsf{E}\left[\sum_i f_i\left(\sum_n a_n h[k-i,n]+v_{k-i}\right)^2 \sum_{i_3} f^*_{i_3}\left(\sum_{n_3} a_{n_3}h[k-i_3,n_3]+v_{k-i_3}\right)\right]$$

and

$$\mathsf{E}\left[\mathbf{f}_k^T \mathbf{r}_k r_{k-m}\right] = \mathsf{E}\big[\quad \textstyle\sum_i f_i \left(\sum_n a_n h[k-i,n] + v_{k-i}\right) \left(\sum_{n_2} a_{n_2} h[k-m,n_2] + v_{k-m}\right)\big]$$

Doing the second term (which is far easier) first, we have

$$\mathsf{E}\left[\mathbf{f}_k^T \mathbf{r}_k r_{k-m}\right] = \mathsf{E}[|a|^2] \sum_n \sum_i h^*[k-i,n] h[k-i,n] f_i + f_m \mathsf{E}[|v_{k-m}|^2]$$

Writing out all of the sums in the first term

$$\begin{aligned}\mathsf{E}\big[\quad & \left(\textstyle\sum_{n_1} \sum_{n_2} \sum_{i_2} \sum_{i_2} a_{n_1} a_{n_2} f_{i_1} f_{i_2} h[k-i_1,n_1] h[k-i_2,n_2]\right. \\ & \left. +2 \textstyle\sum_{i_1} f_{i_1} v_{k-i_1} \sum_{n_2} a_{n_2} \sum_{i_2} f_{i_2} h[k-i_2,n_2] + \sum_{i_1} f_{i_1} v_{k-i_1} \sum_{i_2} f_{i_2} v_{k-i_2}\right) \\ & \left(\textstyle\sum_{n_3} a^*_{n_3} \sum_{i_3} f^*_{i_3} h^*[k-i_3,n_3] + \sum_{i_3} f^*_{i_3} v^*_{k-i_3}\right) \\ & \left(\textstyle\sum_{n_4} a^*_{n_4} h^*[k-m,n_4] + v^*_{k-m}\right)\big]\end{aligned}$$

and expanding, yields

$$\begin{aligned}\mathsf{E}\big[\quad & \Big(\textstyle\sum_{n_1,n_2,n_3} a_{n_1} a_{n_2} a^*_{n_3} \sum_{i_1,i_2,i_3} f_{i_1} f_{i_2} f^*_{i_3} h[k-i_1,n_1] h[k-i_2,n_2] h^*[k-i_3,n_3] \\ & + \textstyle\sum_{n_1} \sum_{n_2} a_{n_1} a_{n_2} \sum_{i_1} \sum_{i_2} \sum_{i_3} f_{i_1} f_{i_2} f^*_{i_3} v^*_{k-i_3} h[k-i_1,n_1] h[k-i_2,n_2] \\ & +2 \textstyle\sum_{n_2} \sum_{n_3} a_{n_2} a^*_{n_3} \sum_{i_1} \sum_{i_2} \sum_{i_3} f_{i_1} f_{i_2} f^*_{i_3} h[k-i_2,n_2] h^*[k-i_3,n_3] v_{k-i} \\ & +2 \textstyle\sum_{i_1} \sum_{i_2} \sum_{i_3} f_{i_1} f_{i_2} f^*_{i_3} v_{k-i_1} v^*_{k-i_3} \sum_{n_2} a_{n_2} \sum_{i_2} h[k-i_2,n_2] \\ & + \textstyle\sum_{i_1} \sum_{i_2} \sum_{i_3} f_{i_1} f_{i_2} f^*_{i_3} v_{k-i_1} v_{k-i_2} v^*_{k-i_3}\Big) \\ & \left(\textstyle\sum_{n_4} a^*_{n_4} h^*[k-m,n_4] + v^*_{k-m}\right)\big]\end{aligned}$$

expanding again produces

$$
\begin{aligned}
\mathsf{E}[\quad & \textstyle\sum_{n_1}\sum_{n_2}\sum_{n_2}\sum_{n_3}\sum_{n_4} a_{n_1}a_{n_2}a^*_{n_3}a^*_{n_4}\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3} \\
& h[k-i_1,n_1]h[k-i_2,n_2]h^*[k-i_3,n_3]h^*[k-m,n_4] \\
& +\textstyle\sum_{n_1}\sum_{n_2}\sum_{n_3} a_{n_1}a_{n_2}a^*_{n_3}\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}h[k-i_1,n_1]h[k-i_2,n_2] \\
& h^*[k-i_3,n_3]v^*_{k-m} \\
& +\textstyle\sum_{n_1}\sum_{n_2}\sum_{n_4} a_{n_1}a_{n_2}a^*_{n_4}h^*[k-m,n_4]\sum_{i_1}\sum i_2\sum i_3 f_{i_1}f_{i_2}f^*_{i_3}v^*_{k-i_3} \\
& h[k-i_1,n_1]h[k-i_2,n_2] \\
& +\textstyle\sum_{n_1}\sum_{n_2} a_{n_1}a_{n_2}\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}v^*_{k-i_3}v^*_{k-m} \\
& h[k-i_1,n_1]h[k-i_2,n_2] \\
& +2\textstyle\sum_{n_2}\sum_{n_3}\sum_{n_4} a_{n_2}a^*_{n_3}a^*_{n_4}\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}v_{k-i_1} \\
& h[k-i_2,n_2]h^*[k-i_3,n_3]h^*[k-m,n_4] \\
& +2\textstyle\sum_{n_2}\sum_{n_4} a_{n_2}a^*_{n_4}\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}v_{k-i_1}v^*_{k-i_3} \\
& h[k-i_2,n_2]h^*[k-m,n_4] \\
& +\textstyle\sum_{n_4} a^*_{n_4}h^*[k-m,n_4]\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}v_{k-i_1}v_{k-i_2}v^*_{k-i_3} \\
& +2\textstyle\sum_{n_2}\sum_{n_3} a_{n_2}a^*_{n_3}\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}v_{k-i_1}v^*_{k-m}h[k-i_2,n_2]h^*[k-i_3,n_3] \\
& +2\textstyle\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}v_{k-i_1}v^*_{k-i_3}v^*_{k-m}\sum_{n_2} a_{n_2}h[k-i_2,n_2] \\
& +\textstyle\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}v_{k-i_1}v_{k-i_2}v^*_{k-i_3}v^*_{k-m}]
\end{aligned}
$$

Now we drop terms which, under standard iid distribution of source symbols and iid uncorrelated noise assumptions, have expectation zero

$$
\begin{aligned}
& \textstyle\sum_{n_1}\sum_{n_2}(\delta[n_1-n_2]\mathsf{E}[|a|^4]+(1-\delta[n_1-n_2])(\mathsf{E}[|a|^2])^2) \\
& \textstyle\sum_{i_1}\sum_{i_2}\sum_{i_3} f_{i_1}f_{i_2}f^*_{i_3}h[k-i_1,n_1]h^*[k-i_3,n_1]h[k-i_2,n_2]h^*[k-m,n_2] \\
& +2\textstyle\sum_{n_1}\mathsf{E}[|a|^2]\sum_{i_1}\sum_{i_2}|f_{i_1}|^2 f_{i_2}\mathsf{E}[|v|^2]h[k-i_2,n_1]h^*[k-m,n_1] \\
& +2\textstyle\sum_{n}\mathsf{E}[|a|^2]f_{k-m}\mathsf{E}[|v_{k-m}|^2]\sum_{i_2}\sum_{i_3} f_{i_2}f^*_{i_3}h[k-i_2,n]h[k-i_3,n] \\
& +\textstyle\sum_{i}|f_i|^2 f_m(\delta[i-m]\mathsf{E}[|v_i|^4]+(1-\delta[i-m])(\mathsf{E}[|v_i|^2])^2)
\end{aligned}
$$

Thus, we have that the total sensitivity is

$$
\begin{aligned}
S_{f_m}(\mathbf{f},\tau,\theta,g) = \; & \gamma \left[\mathsf{E}[|a|^2] \sum_n \sum_i h^*[k-i,n]h[k-i,n]f_i + f_m\mathsf{E}[|v_{k-m}|^2]\right] - \\
& \Big[\sum_{n_1} \sum_{n_2} (\delta[n_1-n_2]\mathsf{E}[|a|^4] \\
& +(1-\delta[n_1-n_2])(\mathsf{E}[|a|^2])^2) \sum_{i_1} \sum_{i_2} \sum_{i_3} f_{i_1} f_{i_2} f^*_{i_3} h[k-i_1,n_1] \\
& h^*[k-i_3,n_1]h[k-i_2,n_2]h^*[k-m,n_2] \\
& +2\sum_{n_1} \mathsf{E}[|a|^2] \sum_{i_1} \sum_{i_2} |f_{i_1}|^2 f_{i_2} \mathsf{E}[|v|^2] h[k-i_2,n_1]h^*[k-m,n_1] \\
& +2\sum_n \mathsf{E}[|a|^2] f_{k-m} \mathsf{E}[|v_{k-m}|^2] \sum_{i_2} \sum_{i_3} f_{i_2} f^*_{i_3} h[k-i_2,n]h[k-i_3,n] \\
& + \sum_i |f_i|^2 f_m \left(\delta[i-m]\mathsf{E}[|v_i|^4] + (1-\delta[i-m])(\mathsf{E}[|v_i|^2])^2\right)\Big]
\end{aligned}
$$

Appendix B

Digital Receiver Component Interaction Comments

This appendix contains some quotes pulled from the adaptive receiver literature about multiple adaptive devices interacting. From these quotes we can discern recognition of the interaction of different adaptive systems to be an existing problem about which very little research has been done. Specifically, adaptive communications receivers are a prime example of this phenomenon. Algorithms are designed and proposed, complete with signal models, which assume that all of the other tasks in the receiver have been performed perfectly. To give some validity to these broad reaching statements, here are some quotes from the adaptive receiver literature

- "The sequential order of some of the digital signal processing blocks can be interchanged, depending on the realization constraint." [23] page 225
- "Phase estimation is performed at symbol rate. Since timing recovery is performed before phase recovery, the timing estimation algorithm must either work (i) with an arbitrary phase error offset or (ii) with a phase estimate (phase-aided timing recovery) or (iii) phase and timing are acquired jointly." [23]
- "Phase recovery can be performed after timing recovery. This is the opposite order as found in classical analog receivers." [23]
- "When deriving a synchronization algorithm from the ML criterion, one as-

sumes an idealized channel model (to be discussed shortly) and constant parameters, at least for quasi-static channels. In principle, a more realistic channel model and time-variable parameters could be taken into account, but it turns out that this approach is mathematically far too complicated. In view of the often crude approximations made to arrive at a synchronizer algorithm, it makes no sense to consider accurate channel models. Instead, one derives synchronization algorithms under ..." [23] page 273

- "Several carrier synchronizers make use of a timing estimate, and several symbol synchronizers make use of a carrier phase estimate. Hence, carrier and symbol synchronizers in general do not operate independently of each other." [23] page 345

- "In the case of a feedback carrier synchronizer which makes use of a timing estimate (the same reasoning holds for a feedback symbol synchronizer which makes use of a carrier phase estimate), the phase error detector characteristic depends not only on the phase error ϕ but also on the timing error e..." [23] page 346

- "..it follows that the linearized tracking performance of the feedback carrier synchronizer is affected by the timing error." [23] page 346

- "Consequently, carrier synchronizers will be analyzed under the assumption of perfect carrier recovery." [23] page 347

- "Both synchronizers make use of the carrier phase estimate and of the receiver's decisions.... it will be assumed that a perfect carrier phase estimate is available." [23] page 351

- "When operating in the tracking mode, the considered carrier and symbol synchronizers do not interact after linearization of the system equations, when the baseband pulse at the matched filter output is real and even." [23] page 377
- "frequency correction has been applied to the signal entering the carrier phase and symbol synchronizers." [23] page 400
- "Several feedback carrier synchronizers make use of a timing estimate, and several feedback symbol synchronizers make use of a carrier phase estimate..." [23] page 401
- "Consequently, in general the dynamics of carrier and symbol synchronizers are coupled. Although for most synchronizers the coupling is negligible in the tracking mode because linearization of the system equations about the stable equilibrium point yields uncoupled dynamics, the coupling can no longer be ignored during acquisition, where estimation errors might be so large that linearization does no longer apply." [23] page 401
- "Acquisition performance is hard to predict, because the analysis of coupled nonlinear dynamical systems that are affected by noise is very difficult..." [23] page 402
- "The coupling between the dynamics of the carrier and symbol synchronizer can be avoided by using a carrier synchronizer which operates independently of the timing estimate [a candidate algorithm is the discrete time version of (6-209)] and a symbol synchronizer which does not use the carrier phase estimate [candidate algorithms are the NDA algorithm (6-150) and the Gardener algorithm (6-205)]" [23] page 402

- "The time needed to acquire both the carrier phase and the symbol timing is the maximum of the acquisition times for the individual synchronization parameters." [23] page 402

- "Let us consider the case where the symbol synchronizer does not use the carrier phase estimate, but the carrier synchronizer needs a timing estimate (this is the case for most carrier synchronizers)." [23] page 402

- "During the symbol timing acquisition, the carrier phase acquisition process is hardly predictable, because it is influenced by the instantaneous value of the symbol timing estimate. However, when the symbol timing acquisition is almost completed the carrier synchronizer uses a timing estimate which is close to the correct timing; from then on, the carrier phase acquisition is nearly the same as for correct timing, and is essentially independent of the symbol synchronizer operation." [23] page 402

- "On the other hand, a coupling is introduced when both the carrier and the symbol synchronizer use DA algorithms, because most feedback DA carrier synchronizers (symbol synchronizers) need a timing estimate (a carrier phase estimate)." [23] page 402

- "In the following, we consider the acquisition of the carrier synchronizer assuming perfect timing; a similar reasoning applies to a symbol synchronizer, assuming perfect carrier phase estimation." [23] page 403

- "The results (6-244) and (6-246) for carrier synchronization assume perfect timing" [23] page 406

- "Because of their long acquisition time, feedback algorithms are not suited for short burst operation." [23] page 407

- "The received burst is processed a first time to derive the symbol timing by means of an NDA algorithm which does not use a carrier phase estimate." [23] page 407
- "The received burst is processed a second time to derive a carrier phase estimate by means of an NDA algorithm which makes use of the timing estimate obtained in the previous step." [23] page 407
- "For the acquisition of the carrier phase and the symbol timing, it is advantageous that there is no strong coupling between the carrier and the symbol synchronizers. In the case of feedforward synchronization, the coupling increases the computation requirements because a two-dimensional maximization over the carrier phase and the symbol timing must be performed." [23] page 416
- "Therefore, it is recommended that at least one synchronizer operates completely independently of the other." [23] page 416
- "The least processing is required when at least one synchronizer operates independently of the other." [23] page 416
- "When the post-processing also performs additional filtering in order to reduce the variance of the individual feedforward estimates, the feedback gives rise to an acquisition transient, and hangup might occur." [23] page 417
- "we assume that timing has been established prior to frequency synchronization." [23] page 478

- "the allowed frequency offset range is now no longer determined solely by the capture range of the frequency estimation algorithm itself but also by the capability of the timing synchronization algorithm to recover timing in the presence of a frequency offset." [23] page 478

- "If this cannot be guaranteed, a coarse frequency estimate of a first stage operating independently of timing information (see Sections 8.2 and 8.3) is required." [23] page 478

- "Timing and phase synchronizer are separated (see Figure 10-15), which avoids interaction between these units. From a design point of view this separation is also advantageous..." [23] page 549

- "Timing recovery was the square and filter algorithm. The algorithm works independently of the phase." [23] page 549

- "The second problem is the interaction between the combiner, the synchronization circuits and the adaptive baseband equalizer." [24] page 1

- "on the contrary, the convergence behavior outlined in [9] is not so good in any distorted channel condition, because of interactions with the equalizer updating and carrier recovery loops." [24] page 4

- "in this case, to avoid interaction between the equalizer control and the carrier recovery loop, a phase independent equalizer is advisable." [24] page 5

- "therefore, the effects of interaction due to simultaneous estimation of data, timing, and carrier could not be studied. These effects are considered in this paper, and the earlier results revisited and compared." [25] page 1

- "the sensitivity of various equalizers to sampling timing phase is then examined. Fading effects on carrier phase estimation are considered. Individual and joint operation (interaction) of synchronization and demodulation circuitry is investigated." [25] page 1

- "The choice of a correct timing phase is critical for the performance of a conventional (synchronous) equalizer." [25] page 4

- "We can see from (13) and (17) that ϕ has an effect on the estimation of δ and vice versa." [25] page 4

- "Timing sensitivity computations on TE's...confirm the well-known result that FSE's are less sensitive to timing variations that synchronous equalizers." [25] page 5

- "during midband fades the T/2 TE appears to be slightly to inferior to the 3T/4 TE in timing sensitivity- a result which may on the surface appear to be contradictory to Gitlin." [25] page 5

- "The T-spaced DFE also shows a higher sensitivity to timing that either of the fractional DFE's... during both MP and NMP fades." [25]

- "Considering the timing sensitivity of the T-DFE, a timing shift of this magnitude results in a deteriorated performance. Fractional DFE's, on the other hand, are more robust to timing deviations from optimum and are able to "tolerate" this shift." [25] page 6

- "The choice of an optimum timing phase criterion cannot be regarded as universal for all equalizers studied." [25] page 7

- ”At a high SNR, the timing-recovery process can be separated from the decoding process with little penalty; timing recovery can use an instantaneous decision device to provide tentative decisions that are adequately reliable, which can then be used to estimate the timing error.” [26] page 1

- ”At low SNR, however, timing recovery and decoding are intertwined. The timing-recovery process must exploit the presence of the code to get reliable decisions, and the decoders must be fed well-timed samples to function properly.” [26] page 1

- ”derives joint ML phase and frequency estimation but the symbol timing is assumed perfectly known. In actual application, however, all these parameters need to be estimated from each data burst.” [27] page 1

- ”Timing recovery is necessary because like the carrier phase, symbol timing is subject to Doppler effects. The compensation could be left to the equalizer [12], which is able to perform this task as long as the corresponding time span of the equalizer is longer than the cumulated time offset during transmission.” [28] page 1

- ”to restrict the tasks of the equalizer to inter-symbol interference (ISI) reduction, time synchronization should be better performed by a separate unit.” [28] page 2

- about DD LMS and DD PLL interaction ”To obtain a good dynamical behavior of the coupled system, the phase synchronizer should be able to track time variations more rapidly than the equalizer to provide the latter with reasonably phase-compensated symbol decisions. Therefore, the step-size parameter α is chosen roughly 10 times the value of μ” [28]

- ”The effectiveness of our proposal relies upon the synergic action of clock recovery and adaptive baseband combining, which allows optimal equalization of the two-ray diversity channel.” [24]

- ”This paper considers two solutions to the problem of frequency offset. Firstly, the equalizer coefficients can be updated on a continuous basis. This method is computationally complex and has limited performance. Secondly, joint optimization of equalizer tap weights and demodulator phase can be achieved by incorporating a phase locked loop in the equalizer.” [29] page 1

- ”We will not address these problems in this paper, and we will assume that the system has perfect synchronization.” [30] page 3

- ”These distortions [multipath] can have disastrous effects on decision directed carrier and timing recovery loops (in particular the timing algorithm, as this relies on the symmetry of the pulse shape).” [31]

- ”Since it has been shown in [1] that a (5,5) complex tap DFE yields satisfactory performance during fading, it is logical to take advantage of the full equalizer structure for carrier and timing recovery.” [31]

- ”In this paper we show [via simulations] that the carrier and timing recovery loops do remain in lock with an acceptable amount of jitter, when they are operated in conjunction with the adaptive DF, in the presence of multipath distortion.” [31]

- ”Due to the complex interaction between the receiver subsystems, the results presented here were generated via computer simulation.” [31]

- ”The insignificant effect of the jitter values obtained is put into perspective by relating them to estimated performance degradation, derived assuming worst-case conditions.” [31]

- ”Finally the effect of the carrier and timing recovery loops on the operation of the DFE is examined when the taps adapt according to the complex LMS algorithm. The results indicate that the nominal sampling time is no longer optimal when the signal is distorted by multipath.” [31]

- ”Both the TR loop and tap values adapt (the CR loop remains relatively constant) to a new sampling location such that the mean-square error is reduced compared to when the optimum tap weights and the nominal sampling time are used.” [31]

- ”This result indicates that the interactive effects of the receiver subsystems tend to improve, rather than degrade, system performance.” [31]

- ”Usually, the clock recovery is carried out before equalization because there is a system instability when it is processed after equalization.” [32]

- ”The main disadvantage of decision-directed [timing recovery] techniques is the delay through the feedfoward filter w between the sampling instant and the computation of the timing error update. The delay degrades the phase-locked loop (PLL) tracking capability. However, in the critical initial acquisition phase, the equalizer input is used to form the timing error update, thus eliminating a large part of this latency.” [33]

- ”To optimize the performance of synchronous equalizers, an appropriate sampling instant must be selected. Various schemes have been proposed for tracking its optimum position during fading events; maximization of the sampled signal energy, tracking of the zero-crossings, and minimizations of the output mean-square-error are among the more widely used. If applied at the quantized output, most of these lead to almost the same near-optimal sampling instant. In addition, it seems that using the equalized signal in this way tends to improve the general robustness of timing recovery schemes in the presence of selective fading." [34]

- "The other crucial aspect of carrier recovery is steady-state performance. As the complexity of high-level modulation techniques increases, so does their sensitivity to phase jitter, and the carrier-recovery loop must provide correspondingly tighter control. For normal propagation conditions, this is achieved by choosing the loop parameters appropriately. However, any signal distortions induced by the multipath fading will increase the loop noise spectral density and its noise bandwidth, and reduce the phase detector gain." [34]

- "As in timing recovery, to improve robustness against selective fading, the signals used for carrier recovery should be derived from the output of any adaptive channel equalization. This presents no problems for IF equalizers, but means that, for baseband implementation, the equalizer (or its feed-foward part in a decision-feedback scheme) will be inside the carrier-recovery loop. Excessive delays must then be avoided to prevent consequent degradations in loop performance." [34]

- "Care also needs to be taken to minimize any interaction betweens he equalizer adaptation algorithm and control of the carrier-recovery loop; selective correlation techniques and constraint of the equalizer reference tap are commonly-adopted measures." [34]

Appendix C

Identities/Preliminaries

C.1 Sum of an Infinite Geometric Series

First we develop the finite form of the sum

$$(1-\beta)\sum_{i=0}^{N}\beta^i = \sum_{i=0}^{N}\beta^i - \sum_{i=1}^{N+1}\beta^i = 1-\beta^{N+1}$$

Using the first and last equation, and dividing both side by $1-\beta$, we have

$$\sum_{i=0}^{N}\beta^i = \frac{1-\beta^{N+1}}{1-\beta} \tag{C.1}$$

For $0 < \beta \leq 1$, we can either replicate the argument above, or take the limit as $N \to \infty$ of the finite sum to obtain:

$$\sum_{i=0}^{\infty}\beta^i = \frac{1}{1-\beta} \tag{C.2}$$

C.2 (Discrete) Bellman Gronwell Identity

The version of the discrete Bellman Gronwell Identity found here is taken from [10]. Let $h_k \geq 0$, $x_k \geq 0$, $x_0 \leq g$, $\forall k \in \{0,1,...\}$, and assume we know:

$$x_{k+1} \leq g + \sum_{i=0}^{k} h_i x_i$$

Then

$$x_{k+1} \leq g\left(\prod_{i=1}^{k}(1+h_i)\right)$$

C.3 Lipschitz Continuity

Definition 6 (Lipschitz Continuity). *A function, $f : \mathcal{D} \to \mathbb{R}^M$, $\mathcal{D} \subseteq \mathbb{R}^N$ is said to be Lipschitz continuous with constant $L > 0$ if for all $x_1, x_2 \in \mathcal{D}$ the function* *satisfies* $\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$.

Theorem 11 (Differentiability and Lipschitz Continuity). *If a function,* $f : \mathcal{D} \to \mathbb{R}^M$*, where* D *is open,* $\mathcal{D} \subseteq \mathbb{R}^N$*, is differentiable on* $\mathcal{D}$ *then it is Lipschitz Continuous on any compact subset* $\mathcal{G}$ *of* $\mathcal{D}$*. Furthermore, the Lipschitz constant obeys* $L \leq \sup_{x \in \mathcal{G}} \left\| \frac{\partial f}{\partial x}(x) \right\|$.

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