Timing Recovery Based on Dispersion Minimization

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Abstract — **This paper investigates the idea of applying "dispersion minimization" to the problem of timing phase recovery, and analyzes the resulting cost surface. The minimization can be readily accomplished via a gradient-style algorithm, and the behavior of the algorithm is analyzed in a variety of situations. In spite of its self-learning adaptation, expressions describing the error surface over which the algorithm evolves show that the error surface is unimodal for received signals which are band limited by the Nyquist frequency for baud and $T/2$ sampling, even in the presence of multipath interference. For the baud sampling of raised cosine pulse shaped signals, the algorithm tends to converge to a unique minimum, except in certain extreme cases. Analysis shows that this algorithm performs as desired in the absence of multipath with raised cosine pulse shaping.**

I. Introduction

Recent efforts to realize digital radio/video broadcasting systems have returned the problem of optimal timing recovery into a significant issue in digital receiver design. One of the features of digital media broadcast channels is long delay spread multipath [1, 11], where the use of fractionally spaced equalizers that are insensitive to timing phase is impractical. The problem of timing recovery consists of estimating the timing frequency and the optimal timing phase. There have been many methods of approaching timing recovery: for instance, standard texts such as [9] and [10] describe common methods of determining the optimal timing offset including the Early-Late Gate method, the minimum Mean-Squared-Error (MSE) method, the Maximum-Likelihood (ML) method, and the Output Energy Maximization (OEM) method. The Early-Late Gate and the ML methods are developed assuming that there are no channel dynamics, and their validity and performance are not obvious in the presence of nontrivial channels. On the other hand, the MSE criterion can be extended to the case where a multipath channel is present, but due to the nature of the algorithm a training sequence or feedback from the decision device is required. Since training sequences reduce the overall system throughput, and since the feedback from a decision device may be unreliable at the synchronization stage, a non-decision-directed (blind) method is preferred.

The output energy maximization method, which can be done blindly, has been analyzed in [3]. A blind method which has not received significant attention uses the idea of dispersion minimization, which is often called the Constant Modulus Algorithm. In [2], Guglielmi et al. considered the constant modulus approach to jointly optimize the combiner weights and timing offsets of a pair of received signals from two antennas. This paper considers the case where the constant modulus algorithm (or the minimization of the dispersion of the received signal) is applied to the timing phase recovery for a single antenna structure but with substantial multipath. This method can be implemented without a training signal, and analysis reveals that the error surface is unimodal in a large variety of situations.

The next section shows how to apply the Dispersion Minimization (DM) idea to the timing recovery problem. Section III analyzes the method by studying the DM cost function, while focusing on signals with pulse shapes that are band-limited to the Nyquist frequency. For such signals, the DM cost function is shown to have a unique global minimum. Section IV studies the DM cost function for raised cosine pulse shaped signals sampled at baud rate. In certain extreme conditions, it is observed that the DM criterion exhibits multimodal local minima, but this does not necessarily obviate the use of the algorithm. Section V calculates the DM cost functions and provides simulations showing the performance of the DM-timing recovery algorithm. The final section concludes.

II. Minimization of Dispersion

Consider a digital communication system, where $\{s_i\}$ and $g_T(t)$ denote the source sequence and pulse shaping filter respectively. In the presence of a multipath channel $c(t)$ and white Gaussian noise $w(t)$, the matched filter output $y(t)$ is given by

$$y(t) = \sum_{i=-\infty}^{\infty} s_i \ \delta(t - iT) \star g_T(t) \star c(t) \star g_R(t) + w(t) \star g_R(t)$$

where T is the baud timing rate and $\star$ represents the convolution operator. Denote the overall impulse response including the pulse shape filter, channel, and matched filter as

$$h(t) := g_T(t) \star c(t) \star g_R(t), \qquad (1)$$

and assume that the matched filter output is sampled at a rate T/L. Then, the sampled output at time $kT/L + \tau$ is given by

$$y(\frac{kT}{L} + \tau) = \sum_{i=-\infty}^{\infty} s_i \ h(\frac{kT}{L} - iT + \tau) + w(t) \star g_R(t)|_{t=\frac{kT}{L}+\tau}.$$

Since the choice of τ does not affect the statistics of the noise term $w(t) \star g_R(t)|_{t=kT/L+\tau}$, the optimal timing offset τ can be chosen independent of the noise without loss of generality.

The problem of timing phase recovery addressed in this paper by the minimization of the dispersion of the samples which is defined as

$$J_{disp} = E\{(|y(kT/L + \tau)|^2 - \gamma)^2\}, \quad \tau \in [-\frac{T}{2L}, \frac{T}{2L}). \qquad (2)$$

[1]This work was supported by Fox Digital.

This can be approximated by the time average

$$J_{avg} = \sum_k (|y_k(\tau)|^2 - \gamma)^2, \tag{3}$$

where the received signal $y_k(\tau) = y(kT/L + \tau)$ is written explicitly as a function of the timing offset τ.

The minimum of J_{avg} with respect to the timing phase τ can be found by differentiating equation (3) with respect to τ, which gives the condition

$$\sum_k (|y_k(\tau)|^2 - \gamma) y_k(\tau) \frac{dy_k(\tau)}{d\tau} = 0. \tag{4}$$

Any optimal sampling time must fulfill equation (4), and the next paragraph shows how to translate this into an implementable form. One can view this algorithm as an application of the conventional CMA algorithm to the problem of timing phase recovery. Figure 1 illustrates a block diagram of the DM timing scheme.

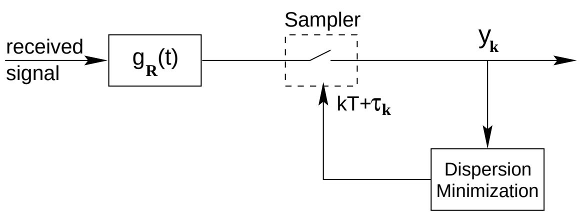


Figure 1: Block Diagram of DM timing algorithm

In order to derive an implementable algorithm, consider the use of an instantaneous stochastic gradient descent method which uses

$$J_{inst} = (|y_k^2(\tau)| - \gamma)^2.$$

As in [4], taking the derivative with respect to τ gives the update equation for the stochastic gradient descent algorithm

$$\tau_{k+1} = \tau_k - \mu \; (|y_k(\tau_k)|^2 - \gamma) \; y_k(\tau_k) \; \frac{dy_k(\tau_k)}{d\tau_k}, \tag{5}$$

where the derivative can be approximated in many ways, perhaps the simplest is:

$$\frac{dy_k(\tau_k)}{d\tau_k} \approx \frac{y_k(\tau_k + \delta) - y_k(\tau_k - \delta)}{2\delta}.$$

Here the stepsize parameter μ may play the role of the averaging (low pass) filter, or a separate filter may also be used.

III. Analysis

This section analyzes the DM based timing phase recovery method by rewriting the cost function J_{disp} of (2) explicitly as a function of the impulse response of the combined channel and shaping filters. Although the DM cost function setting in the previous section is valid for both complex and real signals, for simplicity we consider only real cases without much loss of generality.

A The DM Cost Function

In order to describe the cost function in time domain, first define the L-oversampled samples of $h(t)$ for timing phase τ as an infinite length vector $\mathbf{h}_L(\tau)$, i.e. the $k-$th element of $\mathbf{h}_L(\tau)$, denoted by $\mathbf{h}_L(\tau)_k$, is the sample of $h(t)$ at the time $kT/L + \tau$,

$$\mathbf{h}_L(\tau)_k = h(kT/L + \tau).$$

Similarly, define the samples of $h(t)$ sampled at baud rate for timing phase $l\frac{T}{L}+\tau$ $(l = 0, ...L-1)$ as an infinite length vector $\mathbf{h}(l\frac{T}{L} + \tau)$, i.e.

$$\mathbf{h}(l\frac{T}{L} + \tau)_k = h(kT + l\frac{T}{L} + \tau), \quad l = 0, ..., L-1.$$

Since the $\{s_i\}$ are independent and identically distributed, a straightforward calculation shows

$$\begin{aligned} E\left\{y^2(kT/L+\tau)\right\} &= \frac{m_2}{L}\|\mathbf{h}_L(\tau)\|^2 \\ E\left\{y^4(kT/L+\tau)\right\} &= \frac{m_4 + m_2(3-\kappa_s)}{L}\sum_{l=0}^{L-1}\|\mathbf{h}(l\frac{T}{L}+\tau)\|^4 \\ &\quad -\frac{m_2(3-\kappa_s)}{L}\|\mathbf{h}_L(\tau)\|_4^4, \end{aligned}$$

where $m_2 = E\left\{s_i^2\right\}$ and $m_4 = E\left\{s_i^4\right\}$ denote the second and the fourth moments of the source. Furthermore, $\kappa_s = m_4/m_2^2$ denotes the "kurtosis" of the source, and $\|\cdot\|_4$ denotes the ℓ_4 norm, i.e, $\|\mathbf{h}(\tau)\|_4 = \sqrt[4]{\sum \mathbf{h}(\tau)_k^4}$.

Henceforth, without loss of generality, assume that the source power is normalized, i.e. $m_2 = 1$, and denote the deviation of the kurtosis of the source from that of the Gaussian as η_s, i.e.

$$\eta_s := 3 - \kappa_s.$$

With these conventions, $m_4 + m_2(3-\kappa_s) = 3$. Using the above formulas and notation, the cost function (2) for T/L sampled data can be rewritten as

$$\begin{aligned} J_L(\tau) &= \frac{3}{L}\sum_{l=0}^{L-1}\|\mathbf{h}(l\frac{T}{L}+\tau)\|^4 - \frac{\eta_s}{L}\|\mathbf{h}_L(\tau)\|_4^4 \\ &\quad -2\gamma\|\mathbf{h}_L(\tau)\|^2 + \gamma^2. \end{aligned} \tag{6}$$

Since $\|\mathbf{h}_L(\tau)\|^2$ is a constant with respect to τ when $h(t)$ is band limited by $L/2T$ (see (12) in the proof of Theorem 1 or one can argue this with Parseval's theorem), the choice of γ does not change the location of the minima with respect to τ. Hence, for simplicity of analysis, set $\gamma = 0$ for signals band-limited to Nyquist frequency (in practice γ can be set to minimize excess mean squared error), which further simplifies the cost function to

$$J_L(\tau) = \frac{3}{L}\sum_{l=0}^{L-1}\|\mathbf{h}(l\frac{T}{L}+\tau)\|^4 - \frac{\eta_s}{L}\|\mathbf{h}_L(\tau)\|_4^4. \tag{7}$$

B Unimodality of the DM Timing Algorithm

The DM-timing cost function given in (7) is a single parameter function depending on the pulse shape and the channel. In spite of its intricate time domain appearance, a frequency domain approach can reduce the cost function to a relatively simple form for analysis. This subsection shows that the cost function is either constant or a cosine-like function with a unimodal cost surface when $L = 1$ or $L = 2$, under the assumption that the signals are band-limited by the Nyquist frequency.

Theorem 1 Let $h(t)$ be the combined pulse shape and channel as in (1) which is band limited by $\frac{L}{2T}$. Denote its Fourier transform as $H(f)$, and let $H_2(f) := H \star H(f)$ and $H_4(f) := H \star H \star H \star H(f)$, where $\star$ denotes the convolution operator. Suppose $h(t)$ is sampled at the Nyquist rate T/L. Henceforth, without loss of generality, assume $T = 1$.

For $L = 1$ or $L = 2$ (baud or $T/2$ sampling) the DM timing offset cost function either becomes a constant or has a unique minimum:

i) Baud sampling ($h(t)$ is band-limited by 1/2): The cost function in (7) is given by

$$\begin{aligned} J_{L=1}(\tau) &= -2\eta_s\,|H_4(1)|\cos(2\pi\tau + \angle H_4(1)) \\ &\quad +3H_2^2(0) - \eta_s H_4(0), \end{aligned} \tag{8}$$

where $\angle H_4(1)$ denotes the phase of the complex value $H_4(1)$. When $|H_4(1)| = 0$ the cost function is a constant, otherwise it has a unique minimum at

$$\tau_{DM} = -\frac{1}{2\pi}\angle H_4(1). \tag{9}$$

ii) $T/2$-sampling ($h(t)$ is band-limited by 1): The cost function becomes

$$\begin{aligned} J_{L=2}(\tau) &= 6\,|H_2(1)|^2\cos(4\pi\tau + 2\angle H_2(1)) \\ &\quad -2\eta_s\,|H_4(2)|\cos(4\pi\tau + \angle H_4(2)) \\ &\quad +3(H_2^2(0) + 2\,|H_2(1)|^2) - \eta_s H_4(0). \end{aligned} \tag{10}$$

When both $|H_2(1)|$ and $|H_4(2)|$ are zero, the cost function is a constant. Otherwise, define

$$\begin{aligned} A_H &= 3\,|H_2(1)|^2\sin(2\angle H_2(1)) - \eta_s\,|H_4(2)|\sin(\angle H_4(2)), \\ B_H &= 3\,|H_2(1)|^2\cos(2\angle H_2(1)) - \eta_s\,|H_4(2)|\cos(\angle H_4(2)). \end{aligned}$$

Then from trigonometry identities, it has a unique minimum at

$$\tau_{DM} = \begin{cases} -\frac{1}{4\pi}\tan^{-1}\frac{A_H}{B_H} & \text{if } A_H > 0,\ B_H > 0 \\ -\frac{1}{4\pi}\tan^{-1}\frac{A_H}{B_H} + \frac{1}{4} & \text{if } A_H < 0,\ B_H < 0 \\ \frac{1}{4\pi}\tan^{-1}\frac{A_H}{B_H} & \text{if } A_H > 0,\ B_H < 0 \\ \frac{1}{4\pi}\tan^{-1}\frac{A_H}{B_H} + \frac{1}{4} & \text{if } A_H < 0,\ B_H > 0 \end{cases}$$

Proof: The function $\sum_{k=-\infty}^{\infty} h(t + kT_s)$ is periodic with period T_s for an arbitrary sampling time T_s, and hence can be represented by its Fourier Series as

$$\sum_{k=-\infty}^{\infty} h(t + kT_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} H(\frac{k}{T_s})e^{j2\pi\frac{kt}{T_s}}. \tag{11}$$

Similarly, $\sum_{k=-\infty}^{\infty} h^2(t+kT_s)$ and $\sum_{k=-\infty}^{\infty} h^4(t+kT_s)$ are periodic with period T_s. Since the multiplication in time domain is a convolution in frequency domain,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} h^2(t + kT_s) &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} H_2(\frac{k}{T_s})e^{j2\pi\frac{kt}{T_s}} \\ \sum_{k=-\infty}^{\infty} h^4(t + kT_s) &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} H_4(\frac{k}{T_s})e^{j2\pi\frac{kt}{T_s}}. \end{aligned}$$

When $L = 1$, $h(t)$ is band limited by 1/2 and thus the support of $H(f)$, $H_2(f)$, and $H_4(f)$ are $(-1/2, 1/2)$, $(-1, 1)$, and $(-2, 2)$ respectively. Due to the finite support of $H_0(f)$, all but a few of the terms in (11) are zero. Setting $T_s = 1$ in (11) shows that

$$\|\mathbf{h}_L(\tau)\|^2 = H_2(0) \tag{12}$$

$$\begin{aligned} \|\mathbf{h}_L(\tau)\|_4^4 &= H_4(-1)e^{-j2\pi\tau} + H_4(0) + H_4(1)e^{j2\pi\tau} \\ &= 2\Re\left\{H_4(1)e^{j2\pi\tau}\right\} + H_4(0) \\ &= 2\,|H_4(1)|\cos(2\pi\tau + \angle H_4(1)) + H_4(0) \end{aligned} \tag{13}$$

From the above equations and the general DM cost function in (7), the expression for $J_{L=1}$ in (8) follows directly.

When $L \neq 1$, $h(t)$ is band limited by $L/2$ and thus the support of $H(f)$, $H_2(f)$, and $H_4(f)$ are $(-L/2, L/2)$, $(-L, L)$, and $(-2L, 2L)$. Since $\mathbf{h}(\frac{l}{L} + \tau)$ is sampled at $T_s = 1$,

$$\begin{aligned} \|\mathbf{h}(\frac{l}{L} + \tau)\|^2 &= \sum_{k=1}^{L-1} 2\,|H_2(1)|\cos(2\pi k\tau + j2\pi\frac{l}{L} + \angle H_2(k)) \\ &\quad +H_2(0). \end{aligned} \tag{14}$$

On the other hand, since $\mathbf{h}_L(\tau)$ is sampled at $T_s = 1/L$,

$$\begin{aligned} \|\mathbf{h}_L(\tau)\|_4^4 &= 2L\,|H_4(L)|\cos(2\pi\tau L + \angle H_4(L)) \\ &\quad +LH_4(0). \end{aligned} \tag{15}$$

For $L = 2$, the $\|\mathbf{h}(\frac{l}{L} + \tau)\|^2$ term can be written as

$$\begin{aligned} \|\mathbf{h}(\tau)\|^2 &= H_2(0) + 2\,|H_2(1)|\cos(2\pi\tau + \angle H_2(1)) \\ \|\mathbf{h}(\frac{1}{2} + \tau)\|^2 &= H_2(0) - 2\,|H_2(1)|\cos(2\pi\tau + \angle H_2(1)), \end{aligned}$$

and therefore

$$\begin{aligned} \|\mathbf{h}(\tau)\|^4 + \|\mathbf{h}(\frac{1}{2} + \tau)\|^4 &= 4\,|H_2(1)|^2\cos(4\pi\tau + 2\angle H_2(1)) \\ &\quad +4\,|H_2(1)|^2 + 2H_2^2(0). \end{aligned} \tag{16}$$

By combining (15) and (16), expression for $J_{L=2}$ in (10) is obtained. When $L > 2$ the $\|h(\frac{l}{L} + \tau)\|^2$ term has cosines with more than one frequency, which does not guarantee the unimodality of the cost function J_L. ■

As the cost surface can be represented by a cosine term with a constant, it does not have any saddle points, which curbs the speed of convergence in their vicinity. The successful performance of the DM timing phase in the absence of multipath will be presented in Section A. The next section investigates the DM cost function for signals with excess bandwidth, those which cause aliasing when sampled at the baud rate.

IV. DM TIMING OF BAUD-RATE SAMPLED SIGNALS WITH RAISED COSINE PULSE

Finding the optimal timing offset for raised cosine pulse shaped signals, sampled at the baud rate, is an important issue since a properly chosen timing phase can reduce the noise gain of linear equalizers resulting from spectral nulls caused by aliasing [3, 8, 13]. This section investigates the DM timing offsets for raised cosine pulse shaped signals sampled at the baud rate.

From the general DM-cost function (6), the DM-cost function for baud rate sampling is given by

$$J_{rc} = 3\|\mathbf{h}(\tau)\|^4 - \eta_s\|\mathbf{h}(\tau)\|_4^4 - 2\gamma\|\mathbf{h}(\tau)\|^2 + \gamma^2.$$

In this case, $\|h(\tau)\|$ is not constant and hence the choice of γ can change the dynamics. In equalization applications of CMA, γ is set to the expected dispersion of the source, i.e. $\gamma = m_4/m_2$ [7], and this value is appropriate in the timing algorithm as well. For example, under this setting, $\gamma = 1$ for a BPSK source.

Again setting $T = 1$ (without loss of generality), the same arguments as in the proof of Theorem 1 show that

$$\|\mathbf{h}(\tau)\|^2 = H_2(0) + 2\,|H_2(1)|\cos(2\pi\tau + \angle H_2(1))$$
$$\|\mathbf{h}(\tau)\|_4^4 = H_4(0) + 2\sum_{m=1}^{3} |H_4(m)|\cos(2m\pi\tau + \angle H_4(m)).$$

Normalizing the power of the source, i.e. $m_2 := 1$, gives $\kappa_s = m_4$, $\eta_s = 3 - m_4$, and $\gamma = m_4$. The cost function is

$$\begin{aligned} J_{rc} \;=\;& 6\,|H_2(1)|^2\cos(4\pi\tau + 2\angle H_2(1)) \\ &+4(3H_2(0) - m_4)\,|H_2(1)|\cos(2\pi\tau + \angle H_2(1)) \\ &-2(3 - m_4)\sum_{k=1}^{3}|H_4(k)|\cos(2\pi k\tau + \angle H_4(k)) \\ &+C, \end{aligned} \tag{17}$$

where C denotes the constant term. This cost function does not guarantee the unimodality of the DM timing offset, and it is possible to find severe multipath channels (when used with a raised cosine pulse with rolloff factor close to 1) that produce multimodal error surfaces (for instance, see Example in section V-A.2). However, for small rolloff factors α, due to the suppressed tails of the raised cosine, only a few terms in the cost function (17) tend to dominate. For example, for a rolloff factor $\alpha = 0.5$, in the absence of multipath interference, numerical integration shows that

$$H_2(0) \approx 0.88, \quad |H_2(1)| \approx 0.06$$
$$H_4(0) \approx 0.63,\ |H_4(1)| \approx 0.19,\ |H_4(2)| \approx 0.001,$$
$$|H_4(3)| \approx 5 \times 10^{-48}.$$

Thus, the cost function (17) in general can be approximated by

$$\begin{aligned} J \;\approx\;& 4(3H_2(0) - m_4)\,|H_2(1)|\cos(2\pi\tau + \angle H_2(1)) \\ &-2(3 - m_4)\,|H_4(1)|\cos(2\pi\tau + \angle H_4(1)) + C. \end{aligned} \tag{18}$$

Since $H_2(0) < 1$ and $|H_2(1)|$ is an order of magnitude smaller than $|H_4(1)|$, the $\cos(2\pi\tau + \angle H_4(1))$ term generally dominates (see Example A.2). Thus, for raised cosine pulses with small rolloff-factors, although the theoretical possibility of multimodality cannot be eliminated, the DM timing cost function tends to have unimodal timing offset which relies mainly on the phase of $H_4(1)$.

On the other hand, it is possible to exploit timing offset information from $H_2(1)$ alone, blindly, by maximizing the output power of the sampled signals. In this case,

$$\begin{aligned} E\left\{y(kT+\tau)^2\right\} &= H_2(1)e^{j2\pi\tau} + H_2(0) + H_2(-1)e^{-j2\pi\tau} \\ &= 2\,|H_2(1)|\cos(2\pi\tau + \angle H_2(1)) + H_2(0). \end{aligned}$$

This algorithm, which relies on the phase information of $H_2(1)$, is known as the Output Energy Maximization (OEM) algorithm [3, 10], and generates a unique minima $\tau = -\frac{1}{2\pi}\angle H_2(1)$. Furthermore, it has been shown that the minima of the OEM algorithm minimizes the MSE of an infinite length MMSE equalizer output under certain restricted conditions [3]. However, note that the OEM algorithm cannot be employed when the received signal is bandlimited to the Nyquist frequency, i.e. when aliasing is absent. Whether the difference between $H_4(1)$ and $H_2(1)$ can result in better performance of DM-timing offset in the presence of multipath (for example, lower equalizer output MSE when used with finite length equalizers) is a subject of further study.

V. Examples and Simulations

It is a difficult task to obtain a closed form expression for $H_4(f)$ with arbitrary pulse shapes in the presence of multipath channels. This section models an unknown timing as a pure delay channel and studies the cost function for popular pulse shapes. For multipath channels, a numerical analysis of the cost function shows the behavior of the timing recovery algorithm. Finally, a time domain simulation confirms the analysis.

A Examples of the DM cost function

We assume $T_s = 1$ and consider a BPSK source constellation. Consequently, $\eta_s = 2$ and $m_4 = 1$. In the absence of multipath, the unknown timing phase can be modeled as a pure delay τ_0, and therefore

$$h(t) = \delta(t - \tau_0) \star g_T(t) \star g_R(t).$$

By denoting the Fourier transform of $g_T(t) \star g_R(t)$ as $G(f)$,

$$H(f) = e^{-j2\pi\tau_0 f}G(f),$$
$$H_2(f) = e^{-j2\pi\tau_0 f}G_2(f), \quad H_4 = e^{-j2\pi\tau_0 f}G_4(f).$$

Furthermore, notice that

$$\angle H_2(k) = -2k\pi\tau_0, \quad \angle H_4(k) = -2k\pi\tau_0.$$

Using these, it is possible to calculate the DM cost function for specific pulse shapes.

A.1 Signals band-limited to the Nyquist frequency

Band limited sinc pulse with baud sampling: For a sinc pulse in the absence of multipath, $G(f)$ is given as

$$\begin{aligned} G(f) &= \begin{cases} 1 & f \in (-\frac{1}{2}, \frac{1}{2}) \\ 0 & f \notin (-\frac{1}{2}, \frac{1}{2}) \end{cases} \\ G_2(f) &= \begin{cases} f & f \in (-1, 0] \\ 1 - f & f \in [0, 1) \\ 0 & f \notin (-1, 1) \end{cases} \end{aligned}.$$

Thus

$$H_2(0) = 1, \quad H_4(0) = \frac{2}{3}, \quad H_4(1) = \frac{1}{6}e^{-j2\pi\tau_0}$$

The cost function is therefore

$$J(\tau) = \frac{1}{3}(1 - \cos(2\pi\tau - 2\pi\tau_0)),$$

and hence $\tau_{DM} = \tau_0$ as desired ($\cos(2\pi(\tau_{DM} - \tau_0)) = 1$ and $J(\tau_{DM}) = 0$).

A.2 Baud sampling of Raised cosine pulse

The following examples calculate the DM cost in (17) numerically for various situations. The two dominant terms presented in (18) are plotted in order to observe their contributions to the DM minima. Recall that in (18), the DM cost function was approximated by

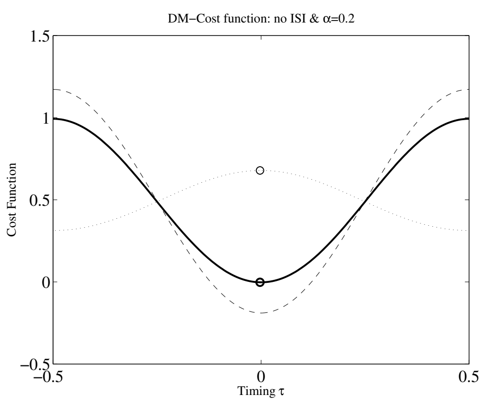
$$\begin{aligned} J &\approx 4(3H_2(0) - m_4)\,|H_2(1)|\cos(2\pi\tau + \angle H_2(1)) \\ &\quad -2(3 - m_4)\,|H_4(1)|\cos(2\pi k\tau + \angle H_4(1)) + C. \end{aligned}$$

The $\cos(2\pi\tau + \angle H_2(1))$ term is depicted by the dotted line and the $\cos(2\pi k\tau + \angle H_4(1))$ term is drawn with a dashed line in the following figures, while the DM cost function is plotted with a solid line. Recall that the maximum of the $\cos(2\pi\tau + \angle H_2(1))$ term is the optimal timing offset of the output energy maximization (OEM) algorithm.

i) In the absence of multipath: From the fact that $\angle H_4(k) = -2k\pi\tau_0$,

$$\cos(2k\pi\tau + \angle H_4(k)) = \cos(2k\pi(\tau - \tau_0)).$$

This implies that for an arbitrary τ_0, the DM cost function with delay τ_0 is the shifted-by-τ_0 version of the DM cost function without delay. Thus, assume $\tau_0 = 0$ in this example (without loss of generality). Figure 2 depicts the DM cost functions for raised cosine pulse with rolloff factor $\alpha = 0.2$. The plot confirms that OEM timing agrees with the DM timing in the absence of ISI.



a) No multipath with $\alpha = 0.2$

Solid : DM cost function, Dotted : $H_2(1)$ term,
Dash : $H_4(1)$ term.
∘ on solid: DM timing, ∘ on dotted: OEM timing

Figure 2: DM cost function under no multipath and raised cosine pulses

ii) Multimodality of the DM cost function: Now consider a severe multipath channel, namely

$$c = \frac{1}{\sqrt{2}}(\delta(t) + 0.95\delta(t - 1.5T)).$$

This channel pulse-shaped with a raised cosine of rolloff factor $\alpha = 1$ has two DM timing minima. As shown in Figure 3-a), the $H_4(1)$ term does not dominate the cost function anymore, and the phase shifted $H_2(1)$ term generates two local minima. The existence of multiple DM timing offsets is not necessarily a shortcoming of the algorithm. Intuitively, for two peaks with the almost same magnitude but with different timing phases, there should be two (almost) optimal timing phases, one for each peak.

However, as the rolloff factor decreases or the multipath diminishes, the $H_4(1)$ term begins to dominate again, and a unique minimum returns. Figure 3 b) and c) present, the DM-cost function for a milder channel $c = \frac{1}{\sqrt{2}}(\delta(t)+0.5\delta(t-1.5))$ with the same rolloff factor $\alpha = 1$ and for the same severe channel ($c = \frac{1}{\sqrt{2}}(\delta(t)+0.95\delta(t-1.5))$) with a smaller rolloff factor ($\alpha = 0.5$) respectively. Interestingly, for the mild channel case DM timing and OEM are located close to each other (Figure 3-b)), while for the smaller rolloff factor and severe channel case they are located in quite different places (Figure 3-c)), although each have a unique timing minimum.

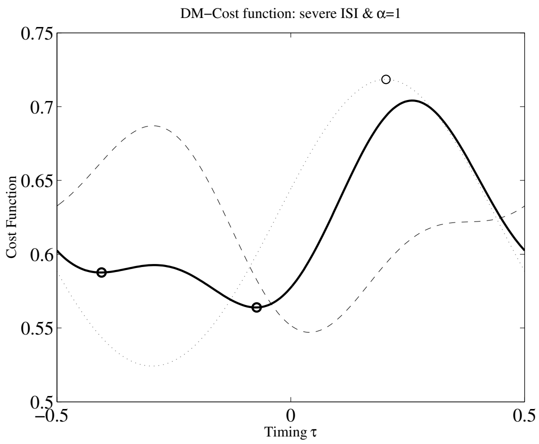
B Simulation

This subsection presents a simulation which confirms the analysis. The simulation is conducted under the same conditions as in Example A.2-ii), i.e. a severe multipath channel $c = \frac{1}{\sqrt{2}}(\delta(t)+0.95\delta(t-1.5))$ and a rolloff factor of $\alpha = 1$. The derivative of sampled output with respect to timing phase $dy_k(\tau_k)/d\tau_k$ was implemented by numerical differentiation using re-sampling. Figure 4 shows the convergence of the DM algorithm to two local minima depending on the initialization after ensemble averaging over 30 different simulations for each initialization.

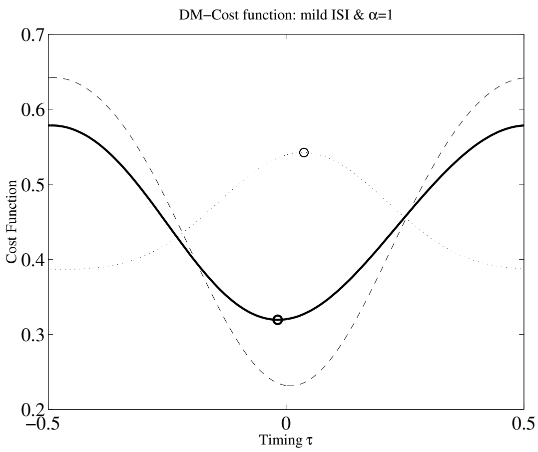
VI. Discussions and Conclusions

This paper has considered a method of timing phase recovery based on the idea of dispersion minimization. There are several kinds of extensions and generalizations of this idea. First, the dispersion function of (2) can be replaced with any one of the generalized CMA forms ($J_{disp} = E\{|y_m^p(\tau) - \gamma|^q\}$ for any p and q). Second, by analogy with equalization, the dispersion minimization method can be used to start the timing recovery, followed by a switch to a decision directed method once reliable symbol decisions are available. Third, the algorithm can be modified in a number of standard ways. For instance, the step size parameter μ can be made time varying (such as the “normalized LMS” method of [6]) for faster convergence at the start and better excess error properties once convergence has been attained. From the implementation viewpoint, there are many methods of approximating the derivative in (5), and each leads to a slightly different algorithm update term. As motivated by Example A-ii, the method can be modified to perform timing phase recovery by maximizing the dispersion instead of minimizing it, depending on the situations in which the algorithm is applied. Finally, small amounts of timing frequency offset can be possibly recovered by tracking the variations in the timing phase. The tracking capability, however will be limited by the shallowness of the cost surface at the location of the minima, step size and excess MSE. Since a tracking lag is invariably present a unimodal cost surface is desirable as this precludes the possibility of oscillating between the various local minima.

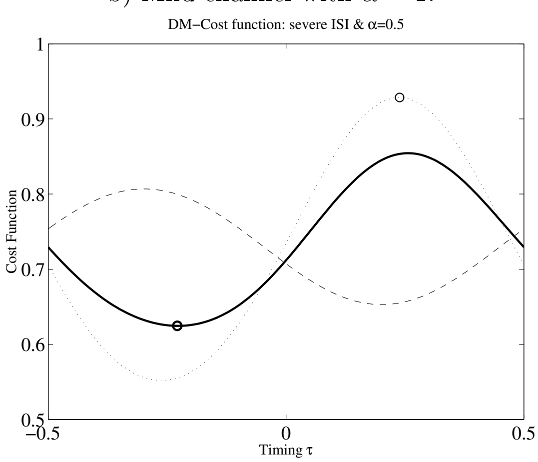
We have focused on a general analysis of the DM timing phase cost function in various settings, and have shown that



a) Severe channel with $\alpha = 1$.



b) Mild channel with $\alpha = 1$.



c) Severe channel with $\alpha = 0.5$.

Solid : DM cost function, Dotted : $H_2(1)$ term,
Dash : $H_4(1)$ term.
∘ on solid: DM timing, ∘ on dotted: OEM timing

Figure 3: Raised cosine rolloff $\alpha = 1$ under severe multipath channel

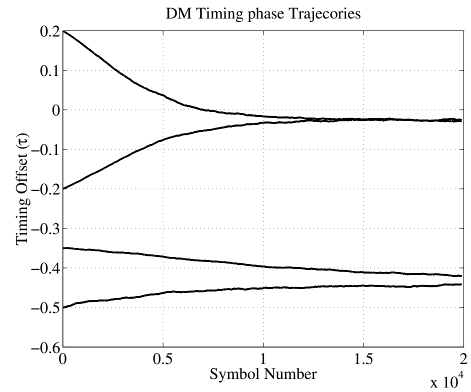


Figure 4: Parameter Trajectories of DM algorithm

the cost function bears a unique global minimum for signals band limited by the Nyquist frequency up to $T/2$ sampling. However, the unimodality does not hold in general for baud sampling of excess bandwith signals. In the absence of multipath, in both cases (i.e., under and over Nyquist sampling) this algorithm produces a unique minimum which recovers timing phase mismatch. In the presence of multipath, the performance of DM timing phase selection is an important area for further investigation.

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