Bidirectional Decision Feedback Equalizer: Infinite Length Results

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Abstract

The use of a decision feedback equalizer (DFE) that employs time-reversal of the received sequence in conjunction with a normal mode operation is considered for a packet transmission system. The bidirectional decision feedback equalizer (BiDFE) combines the output of the normal mode and time-reversal mode DFEs and is known to provide improved performance. This paper shows that, under the decision-error-free assumption, the BiDFE, with an infinite length MMSE-DFE in each stream, has a smaller performance penalty from the matched filter bound (MFB) when compared to each individual stream. The BiDFE tap coefficients are optimized to minimize the overall MSE and it is shown that the infinite length MMSE-BiDFE attains the MFB.

1 Introduction

The use of packet based communication systems, for example GSM, has made block processing of the received data feasible and sometimes necessary. In such a scenario anti-causal processing becomes a possibility and in some cases may be advantageous. Bidirectional equalization of the received signal was first considered in [1, 2]. The use of selective time-reversal for a decision feedback equalizer (DFE) was described in [2, 3], in which, Ariyavisitakul proposes the use of two parallel DFE structures, one for the received sequence and the other for the time-reversed version of the received sequence. A time-reversal operation is done by reversing the sequential order of the received samples, in time, prior to equalization. As a result, the equivalent channel impulse response as seen by the equalizer becomes a time-reverse of the actual channel impulse response. This results in an inversion of the root locations of the channel impulse response, i.e., the minimum phase roots become maximum phase roots and vice-versa. When a finite length DFE is used, the performance is usually different for the normal mode DFE and the time-reversal mode DFE, and hence, selecting the stream with lesser MSE is beneficial. However, the performance can be further improved by combining the two streams rather than selecting one.

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This combinatorial approach was proposed in [4], in which the authors, based on a reconstruction and arbitration technique, exploit the diversity in the error bursts between the normal mode DFE outputs and the time-reversal mode DFE outputs to improve performance. Error propagation, an impairment of the DFE caused by incorrect decision feedback to the feedback filter (FBF), is causal and hence the error burst for a normal mode DFE and the time-reversal mode DFE proceed in opposite directions in time.

However, the advantage of combining the normal mode and the time-reversal mode DFE outputs is not limited to mitigating error propagation. In fact, even in the presence of ideal feedback, namely error-free-decision feedback, the noise at the output of the normal mode DFE and the time-reversal mode DFE exhibit a low correlation. This results in a smaller value of noise-enhancement for the composite structure, namely the bidirectional decision feedback equalizer (BiDFE), that combines the normal mode DFE and the time-reversal mode DFE outputs, when compared to either of the two constituent DFEs [5]. This "diversity" arises from the assumption that the past symbols are known to the normal mode DFE and the future transmitted symbols are known to the time-reversal mode DFE. The anti-causal processing and the nonlinear structure of the DFE make this knowledge, although imperfect in the presence of decision errors, possible.

In [3], the SNR performance loss suffered by an infinite length DFE when compared with the matched filter bound (MFB) was evaluated. The MFB, also known as the ISI-free bound, is defined as the SNR that can be obtained with a matched filter at the receiver [6], provided all the past/future interfering symbols are perfectly known or if only the current symbol is transmitted. The MFB provides an upper bound on the SNR that can be obtained by any equalizer structure. This paper evaluates the reduction in performance penalty from the MFB of the infinite length BiDFE when compared to an infinite length DFE. In addition, the infinite length BiDFE filter coefficients are

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optimized to minimize the overall MSE of the structure, rather than the MSE of the individual streams, and the tap optimized BiDFE is shown to attain the MFB.

2 System Model

Consider the transmission of a data block of size N symbols through a digital base-band channel with a finite-impulse response (FIR). The received sequence $r(n)$ is

$$r(n) = \sum_{k=0}^{L} c(k)s(n-k) + w(n) \tag{1}$$

where $\mathbf{c} = [c(0)\ c(1)\ \ldots\ c(L)]^T$ is the channel impulse response with $L+1$ taps, $w(n)$ is the additive noise sequence, and $s(n)$ is the transmitted source sequence. The time-reversal of the received block of data can be expressed as

$$\tilde{r}(n) \triangleq r(N-n+1) = \sum_{k=-L}^{0} c(-k)\tilde{s}(n-k) + \tilde{w}(n) \ , \tag{2}$$

where *tilde* implies time-reversal. This is equivalent to transmitting the time-reversed source sequence through a channel whose coefficients are the time-reversed version of the channel $\mathbf{c}$.

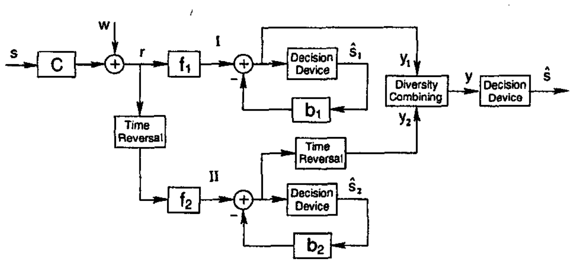

Figure 1: Structure of a Bidirectional DFE.

The block diagram of the BiDFE receiver structure, proposed in [5], is illustrated in Figure 1. The received sequence $r(n)$ is equalized in stream I using a DFE with coefficients $[\mathbf{f_1}, \mathbf{b_1}]$. The sequence $r(n)$ is time-reversed in stream II and equalized using another DFE with coefficients $[\mathbf{f_2}, \mathbf{b_2}]$. The soft outputs of the normal mode and the time-reversal mode DFE are combined using a diversity combiner. In this paper, a memoryless linear combiner will be assumed, i.e., $y(n) = \lambda y_1(n) + (1-\lambda)y_2(n)$. The combiner coefficient λ is optimized (the optimal value of λ is derived in [5]) to minimize the MSE at the output $y(n)$.

In this paper we make the following assumptions.

- The source sequence $s(n)$ is i.i.d with unit variance.
- The noise sequence $w(n)$ is zero mean, white with variance σ_w^2, and is uncorrelated to the source sequence.
- The channel coefficients are unit norm, i.e., $\sum_k |c(k)|^2 = 1$ and the channel does not have a spectral null.
- The decisions that are fed back to the FBF of the DFE are correct, i.e., ideal feedback.

3 Performance of Infinite Length BiDFE

The infinite length MMSE-DFE tap coefficients were derived in [7]. Consider the use of an MMSE-DFE for each stream of the BiDFE structure illustrated in Figure 1. In this section, we evaluate the performance improvement provided by the infinite length BiDFE over an MMSE-DFE, by comparing the performance loss of these two equalizers from the MFB. The MFB is defined as (see [6]),

$$\gamma_{MFB} \triangleq \sum_k \frac{|c(k)|^2}{\sigma_w^2} \ . \tag{3}$$

Let the Z-transform of the channel impulse response be

$$C(z) = c(0)\prod_{i=1}^{L}(1-\alpha_i z^{-1}) \ , \tag{4}$$

where α_i are the root locations of the channel impulse response polynomial. Then, the Z-transform of the time-reversed channel impulse response $\tilde{C}(z)$ is,

$$\begin{aligned} \tilde{C}(z) &\triangleq C(z^{-1}) \ , \\ &= c(0)\prod_{i=1}^{L}(1-\alpha_i z) \ , \\ &= c(L)z^{L}\prod_{i=1}^{L}\left(1-\frac{z^{-1}}{\alpha_i}\right) \ . \end{aligned} \tag{5}$$

In equation (5), the second equality follows from

$$\frac{c(L)}{c(0)} = \prod_{i=1}^{L}\alpha_i \ . \tag{6}$$

Under the high SNR assumption, the feedforward filter (FFF) for the normal mode and the time-reversal operations are given by

$$F_1(z) \approx \frac{1}{c(0)} \prod_{i=1,|\alpha_i|>1}^{L} \frac{(\alpha_i^* - z^{-1})}{\alpha_i^*(1-\alpha_i z^{-1})} \ , \tag{7}$$

and

$$F_2(z) \approx \frac{z^{-L}}{c(L)} \prod_{i=1,|\alpha_i|<1}^{L} \frac{\alpha_i(1-\alpha_i^* z^{-1})}{(\alpha_i - z^{-1})} \ . \tag{8}$$

The feedback filters $B_1(z)$ and $B_2(z)$ are chosen such that the post-cursor ISI is perfectly canceled. In [3], Ariyavisitakul has shown that both the normal-mode DFE and the time-reversal mode DFE, under the ideal feedback assumption, have the same MSE performance. Furthermore, it was demonstrated that each MMSE-DFE stream suffers from a performance loss from the MFB and output SNR of the MMSE-DFE was shown to be

$$\gamma_{\text{DFE}} = \frac{c(0)c^*(0)}{\sigma_w^2}\left(\prod_{i=1,|\alpha_i|>1}^{L} \alpha_i\alpha_i^*\right)^{-1} . \quad (9)$$

As the two streams yield the same MSE, an equal gain combining scheme, i.e., $\lambda = \frac{1}{2}$, is optimal (see [5]) for the diversity combining block of Figure 1. The overall MSE of the BiDFE is

$$\begin{aligned}\text{MSE}_{\text{BiDFE}} &= E[|y(n) - s(n)|^2] \\ &= E\left[\left|\sum_k \frac{\{f_1(k) + f_2(-k)\}\, w(n-k)}{4}\right|^2\right] .\end{aligned} \quad (10)$$

Let us define $[C(z)]_0$ to represent the constant term in the polynomial expansion of $C(z)$, namely $c(0)$. Then, the MSE can be evaluated in the frequency domain as

$$\text{MSE}_{\text{BiDFE}} = \sigma_w^2 \left[F_{eq}(z)F_{eq}^*(z^{-1})\right]_0 \quad (11)$$

where

$$F_{eq}(z) = \frac{F_1(z) + F_2(z^{-1})}{2} . \quad (12)$$

The performance gain provided by the BiDFE over the DFE is given by,

$$\frac{\gamma_{\text{BiDFE}}}{\gamma_{\text{DFE}}} = \left[\frac{F_1(z)F_1^*(z^{-1})}{F_{eq}(z)F_{eq}^*(z^{-1})}\right]_0 . \quad (13)$$

Equation (13) can be further simplified as

$$\frac{\gamma_{\text{BiDFE}}}{\gamma_{\text{DFE}}} = \frac{2}{1 + \Re\{g(0)\}} , \quad (14)$$

where

$$\begin{aligned}G(z) = \frac{c(0)z^L}{c(L)} &\prod_{i=1,|\alpha_i|<1}^{L} \frac{\alpha_i(\alpha_i^* - z^{-1})}{(1 - \alpha_i z^{-1})} \\ &\times \prod_{i=1,|\alpha_i|>1}^{L} \frac{\alpha_i^*(1 - \alpha_i z^{-1})}{(\alpha_i^* - z^{-1})} .\end{aligned} \quad (15)$$

To illustrate the performance improvement that can be obtained by the use of a BiDFE, we consider two test cases, namely, a 3-tap real symmetric channel and a 3-tap real asymmetric channel. Figure 2 illustrates the gap from the MFB for the conventional DFE and the BiDFE, for a symmetric channel with root locations at α and $\frac{1}{\alpha}$. It is clear that as the channel becomes severe, i.e., root locations closer to the unit circle, the performance gain provided by the BiDFE over the conventional DFE is more than 3 dB. Figure 3 illustrates similar performance improvements for the BiDFE, when applied to a 3-tap asymmetric channel with root locations at α and $\frac{1}{2}$. Although the BiDFE performs better than the MMSE-DFE, it still suffers from a performance penalty when compared to the MFB.

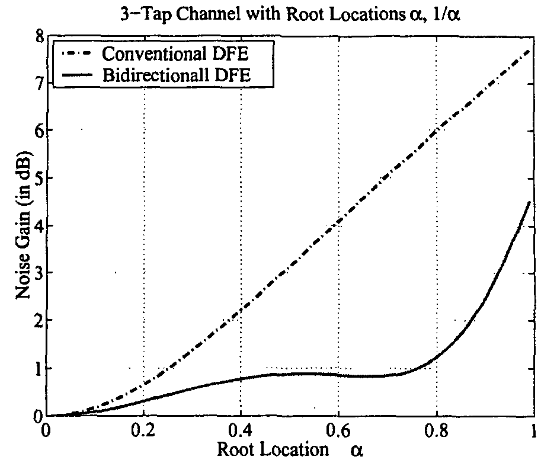

Figure 2: Performance gap from the matched filter bound for a symmetric 3-tap channel with root locations at $(\alpha,\ \frac{1}{\alpha})$.

4 BiDFE Tap Optimization

In the analysis of Section 3 and in the examples corresponding to Figures 2 and 3, the DFE coefficients of each stream of the BiDFE are optimized independently. i.e., an MMSE-DFE setting is chosen. On the other hand, if the DFE filter settings were to be optimized to minimize the overall MSE of the BiDFE and not merely the MSE of each individual stream, it would be possible to further decrease the gap from the MFB. The resulting MSE of the tap-optimized infinite length BiDFE would provide a bound on the potential performance improvements that can be achieved with this structure. It should be noted that the finite length results, which are more useful in practice, asymptotically converge to the infinite length results. The optimization problem consists of determining the BiDFE tap coefficients, namely $[\mathbf{f_1}, \mathbf{b_1}]$, $[\mathbf{f_2}, \mathbf{b_2}]$, and the coeffi-

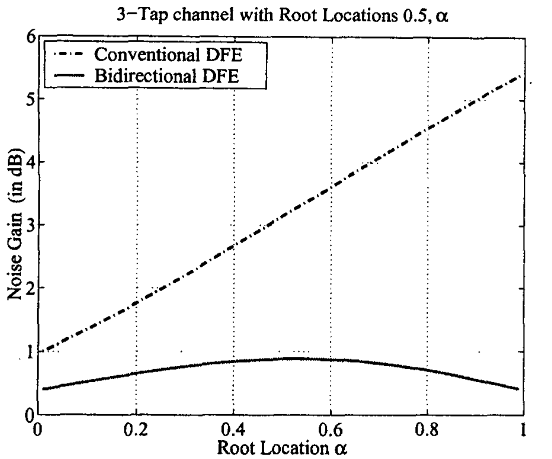

Figure 3: Performance gap from the matched filter bound for an asymmetric 3-tap channel with root locations at $(\alpha, \frac{1}{2})$.

cient of the diversity combiner, λ, that minimizes the overall MSE of the BiDFE.

Theorem 1 *The infinite length MMSE-BiDFE, under the ideal feedback assumption, attains the MFB.*

Proof: Consider the BiDFE receiver structure illustrated in Figure 4. The received signal $r(n)$ is processed using a front-end filter matched to the channel impulse response. The resulting effective impulse response is conjugate symmetric and is given by

$$\Phi(z) = C(z)C^*(z^{-1}) \cdot \tag{16}$$

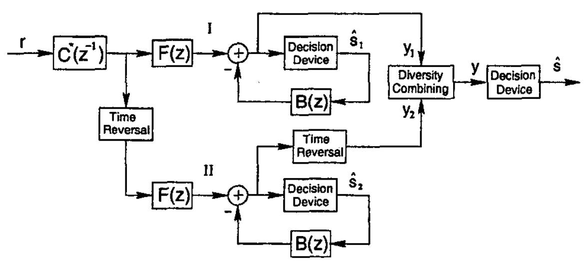

Figure 4: Tap Optimization of Infinite Length BiDFE.

Since, the noise sequence $w(n)$ is assumed to be white, the auto-correlation function of the filtered noise has a Z-transform $\Phi(z)$. Assume that the DFE is of infinite length and is unbiased. Let $[F(z), B(z)]$ denote the DFE tap coefficients of the normal mode DFE. As the time-reversal of $\Phi(z)$, due to its conjugate symmetry property, is $\Phi^*(z)$, a suitable choice for the time-reversal mode DFE is $[F^*(z), B^*(z)]$. Clearly, the DFE structures of the two streams yield the same MSE, and hence an equal-gain combiner, i.e., a choice of $\lambda = \frac{1}{2}$, is optimal (see [5]).

The MSE minimization problem can now be cast in the form

$$\min_{F,B} E\left[|y(n) - s(n)|^2\right] \cdot \tag{17}$$

Let us constrain the FBF to remove all the post-cursor ISI. Then,

$$F(z)\Phi(z) = U(z) + 1 + B(z) \tag{18}$$

where $U(z)$ is purely anti-causal and represents the residual pre-cursor ISI. The MSE of the BiDFE is then given by

$$\begin{aligned}\text{MSE} = &\frac{1}{4}\left[\{U(z) + U^*(z^{-1})\}^2\right]_0 \\ &+ \frac{\sigma_w^2}{4}\left[\Phi(z)\{F(z) + F^*(z^{-1})\}^2\right]_0 \cdot\end{aligned} \tag{19}$$

Since $U(z)$ is purely anti-causal, the first term can be simplified to

$$\frac{1}{4}\left[\{U(z) + U^*(z^{-1})\}^2\right]_0 = \frac{1}{2}\left[U(z)U^*(z^{-1})\right]_0 \tag{20}$$

and attains a minimum value of zero, if and only if $U(z) = 0$. By using equation (18) and the conjugate symmetry property of $\Phi(z)$, the second term of equation (19) can be rewritten as

$$\begin{aligned}&\frac{\sigma_w^2}{4}\left[\Phi(z)\{F(z) + F^*(z^{-1})\}^2\right]_0 = \\ &\frac{\sigma_w^2}{4}\left[\frac{\{2 + B(z) + B^*(z^{-1}) + U(z) + U^*(z^{-1})\}^2}{\Phi(z)}\right]_0 \cdot\end{aligned} \tag{21}$$

Let us define

$$V(z) \triangleq 1 + \frac{B(z) + B^*(z^{-1}) + U(z) + U^*(z^{-1})}{2} \cdot \tag{22}$$

Then, equation (21) simplifies to

$$\frac{\sigma_w^2}{4}\left[\Phi(z)\{F(z) + F^*(z^{-1})\}^2\right]_0 = \sigma_w^2\left[\frac{\{V(z)\}^2}{\Phi(z)}\right]_0 \cdot \tag{23}$$

Lemma 1 If $[V(z)]_0 = 1$ and $[C(z)C^*(z^{-1})]_0 = 1$, then

$$\left[\frac{V(z)V^*(z^{-1})}{C(z)C^*(z^{-1})}\right]_0 \geq 1 \quad (24)$$

and equality is attained if and only if $V(z) = C(z)C^*(z^{-1})$.

Proof: The term $[V(z)]_0$ can be evaluated using the well-known identity based on convolution integral around the unit circle, i.e.,

$$[V(z)]_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(e^{j\theta})d\theta \ \cdot \quad (25)$$

Applying Cauchy-Schwartz inequality to the two continuous functions, $\frac{V(e^{j\theta})}{C(e^{j\theta})}$ and $C^*(e^{j\theta})$, we have

$$\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{V(e^{j\theta})}{C(e^{j\theta})}C(e^{j\theta})d\theta\right]^2 \leq \left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{V(e^{j\theta})V^*(e^{j\theta})}{C(e^{j\theta})C^*(e^{j\theta})}d\theta\right] \times \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} C(e^{j\theta})C^*(e^{j\theta})d\theta\right], \quad (26)$$

where equality is attained if and only if $\frac{V(e^{j\theta})}{C(e^{j\theta})} \propto C^*(e^{j\theta})$, or equivalently, $\frac{V(z)}{C(z)} \propto C^*(z^{-1})$. Since $[V(z)]_0 = 1$ and $[C(z)C^*(z^{-1})]_0 = 1$, equation (26) reduces to

$$\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{V(e^{j\theta})V^*(e^{j\theta})}{C(e^{j\theta})C^*(e^{j\theta})}d\theta\right] \geq 1 \quad (27)$$

with equality if and only if $V(z) = C(z)C^*(z^{-1})$. □

From Lemma 1, the second term in equation (19) is minimized when $V(z) = \Phi(z)$. Hence, the MSE of equation (19) attains a minimum value when $U(z) = 0$ and

$$1 + \frac{B(z) + B^*(z^{-1})}{2} = \Phi(z) \ \cdot \quad (28)$$

As the feedback filter $B(z)$ is purely causal, the optimal tap coefficients are given by,

$$b^{opt}(k) = 2\phi(k)\ , \qquad \forall\, k = 1, 2, \cdots, L \quad (29)$$

and the optimum FFF is

$$F^{opt}(z) = \frac{1 + B^{opt}(z)}{\Phi(z)} \ \cdot \quad (30)$$

From equations (24), (28) and (30), the minimum MSE attained by this choice of BiDFE coefficients is

$$\text{MSE}_{\text{MMSE-BiDFE}} = \sigma_w^2 \quad (31)$$

and hence the maximum output SNR is

$$\gamma_{\text{MMSE-BiDFE}} = \frac{1}{\sigma_w^2}\ , \quad (32)$$

which is same as the MFB. In other words, the MSE optimized infinite length BiDFE attains the MFB. ■

5 Conclusions

The penalty from the matched filter bound for an infinite length BiDFE, with DFE filter settings optimized to minimize MSE of each stream, is computed and is shown to be smaller than the penalty of a MMSE-DFE. The BiDFE tap settings are then optimized to minimize the MSE of the overall structure and it is demonstrated that the optimal MMSE-BiDFE attains the matched filter bound. This illustrates that a BiDFE, under the ideal feedback assumption, offers the best possible performance by any equalizer structure. Although in practice, only finite length filter realizations are possible, the performance of the finite length BiDFE will asymptotically approach the infinite length BiDFE performance. In addition to scaling the gap from the MFB, the BiDFE which is known for mitigating the effects of error propagation may achieve near-optimal performance, if suitable combining schemes, capable of exploiting the diversity between the error bursts caused by the two component DFEs, are employed.

References

[1] H. Suzuki, "Performance of a new adaptive diversity-equalization for digital mobile radio," *Electron. Letters*, vol. 26, no. 10, pp. 626–627, May 1990.

[2] S. Ariyavisitakul, "A decision-feedback equalizer with selective time-reversal operation for high-rate indoor radio communication," in *Proc. GLOBECOM '90*, Dec. 1990, pp. 2035–2039.

[3] S. Ariyavisitakul, "A decision feedback equalizer with time-reversal structure," *IEEE J. Sel. Areas Communication*, vol. 10, no. 3, pp. 599–613, Apr. 1992.

[4] C. S. McGahey, A. C. Singer, and U. Madhow, "Bad: A bi-directional arbitrated decision feedback equalizer," in *Proc. of CISS 2000*, Princeton, NJ, USA, Mar. 2000.

[5] J. Balakrishnan and C. R. Johnson, Jr, "Time-reversal diversity in decision feedback equalization," in *Proc. of Allerton Conf. on Comm. Control and Computing*, Monticello, IL, USA, Oct. 2000.

[6] J. G. Proakis, *Digital Communications*, McGraw-Hill, NewYork, NY, 1995.

[7] P. Monsen, "Feedback equalization for fading dispersive channels," *IEEE Trans. Information Theory*, vol. 17, pp. 56–64, Jan. 1971.